Advanced Engineering Mathematics IV

Mathematics IV deals with the ***Fourier series***, ***Fourier integrals***, ***Fourier transforms*** and the ***Laplace transforms***. The Fourier series are used for the analysis of the periodic phenomena, which often appear in physics and engineering. The Fourier integrals and Fourier transforms expanded the ideas and techniques of the Fourier series to the non-periodic phenomenon. The Laplace transforms are useful for solving ordinary differential equations and initial value problems and some partial differential equations.

1. Review of Ordinary Differential Equations

In this section, the solutions of ordinary differential equations, especially linear ordinary differential equations are reviewed to compare with the solutions by the Laplace transforms to learn later.

1.1 Linear Differential Equations

1.1.1 Homogeneous Linear Differential Equations

Let $y^{(n)}(x) = d^n y(x)/dx^n$ be the nth derivative of the function $y(x)$ and $y'(x) = dy(x)/dx$. The equation is called **linear** if it can be written

$$y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \cdots + p_1(x)y'(x) + p_0(x)y(x) = r(x) \qquad (1.1.1)$$

where $r(x)$ on the right and the coefficients $p_0(x), p_1(x), \cdots, p_{n-1}(x)$ are any given function of x.

If $r(x) \equiv 0$, Eq.(1.1.1) becomes

$$y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \cdots + p_1(x)y'(x) + p_0(x)y(x) = 0 \qquad (1.1.2)$$

and it called **homogeneous**. If $r(x)$ is not identically zero, the equation is called **nonhomogeneous**.

Superposition Principle or Linearity Principle

For the **homogeneous** linear differential equation (1.1.2), sums and constant multiples of solutions on some open interval *I* are again solutions of Eq.(1.1.2) on *I*.

General Solution of Homogeneous Linear Differential Equation

A **general solution** of Eq.(1.1.2) on an open interval *I* is a solution of Eq.(1.1.2) on *I* of the form

$$y(x) = c_1 y_1(x) + \cdots + c_n y_n(x) \qquad (c_1, \cdots, c_n \text{ arbitrary}) \qquad (1.1.3)$$

where $y_1(x), \cdots, y_n(x)$ is a **basis** (or **fundamental system**) of solution of Eq.(1.1.2) on *I* ; that is , these solutions are linearly independent on *I*.

A **particular solution** of Eq.(1.1.2) on *I* is obtained if we assign specific values to the *n* constants $c_1, \cdots, c_n$ in Eq.(1.1.3).

Linear Independence

n functions $y_1(x), \cdots, y_n(x)$ are called **linearly independent** on some interval *I* where they are defined if the equation

$$k_1 y_1(x) + \cdots + k_n y_n(x) = 0 \quad \text{on } I \qquad (1.1.4)$$

implies that all $k_1, \cdots, k_n$ are zero.

These functions are called **linearly dependent** on *I* for some $k_1, \cdots, k_n$ not all zero.

Suppose that the coefficients $p_0(x), p_1(x), \cdots, p_{n-1}(x)$ of Eq.(1.1.2) are continuous on some open interval *I*. Then *n* solutions $y_1(x), \cdots, y_n(x)$ of Eq.(1.1.2) on *I* are linearly dependent on *I* if and only if their **Wronskian**

$$W(y_1, \cdots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \qquad (1.1.5)$$

is zero for some $x = x_0$ in *I*. Furthermore, if $W = 0$ for $x = x_0$, then $W \equiv 0$ on *I*; hence if there is an x_1 in *I* at which $W \neq 0$, then $y_1(x), \cdots, y_n(x)$ are linearly independent on *I*.

1.1.2 Homogeneous Linear Differential Equation with Constant Coefficients

We show how to solve *n*th order homogeneous linear differential equation with real constant coefficients in the form

$$y^{(n)}(x)+a_{n-1}y^{(n-1)}(x)+\cdots+a_1y'(x)+a_0y(x)=0 \qquad (1.1.6)$$

where $y^{(n)}(x)=d^n y(x)/dx^n$ is the highest derivative.

Characteristic Equation and General Solution

By substitution of $y=e^{\lambda x}$ and its derivatives we obtain the **characteristic equation**

$$\lambda^n+a_{n-1}\lambda^{n-1}+\cdots+a_1\lambda+a_0=0 \qquad (1.1.7)$$

of Eq.(1.1.6).

(1) If Eq.(1.1.7) has *n* unequal roots $\lambda_1,\cdots,\lambda_n$, then the *n* solutions

$$y_1(x)=e^{\lambda_1 x},\cdots,y_n(x)=e^{\lambda_n x} \qquad (1.1.8)$$

constitute a basis of all x, and the corresponding general solution of Eq.(1.1.6) is

$$y(x)=c_1e^{\lambda_1 x}+\cdots+c_ne^{\lambda_2 x} \qquad (c_1,\cdots,c_n \text{ arbitrary}) \qquad (1.1.9)$$

(2) If complex roots occur, they must occur in conjugate pairs since the coefficients of Eq.(1.1.6) are real. Thus, if $\lambda=\gamma+i\omega$ is a simple root of Eq.(1.1.7), so is the conjugate $\bar{\lambda}=\gamma-i\omega$, and two corresponding linearly independent solutions are

$$y_1(x)=e^{\gamma x}\cos\omega x,\ y_2(x)=e^{\gamma x}\sin\omega x \qquad (1.1.10)$$

(3) If λ is a **root of order *m***, then *m* corresponding linearly independent solutions are

$$e^{\lambda x},\ xe^{\lambda x},\cdots,x^{m-1}e^{\lambda x} \qquad (1.1.11)$$

(4) If $\lambda=\gamma+i\omega$ is a complex multiple root of order *m*, so is the conjugate $\bar{\lambda}=\gamma-i\omega$. Corresponding linearly independent solutions are

$$e^{\gamma x}\cos\omega x,\ e^{\gamma x}\sin\omega x,\ xe^{\gamma x}\cos\omega x,\ xe^{\gamma x}\sin\omega x,\cdots,\ x^{m-1}e^{\gamma x}\cos\omega x,\ x^{m-1}e^{\gamma x}\sin\omega x \qquad (1.1.12)$$

1.1.3 Nonhomogeneous Linear Differential Equations

We write nonhomogeneous linear differential equations in the standard form

$$y^{(n)}(x)+p_{n-1}(x)y^{(n-1)}(x)+\cdots+p_1(x)y'(x)+p_0(x)y(x)=r(x) \qquad (1.1.13)$$

where $y^{(n)}(x)=d^n y(x)/dx^n$ is the first term and $r(x)$ is not identically zero.
We also need the corresponding homogeneous equation

$$y^{(n)}(x)+p_{n-1}(x)y^{(n-1)}(x)+\cdots+p_1(x)y'(x)+p_0(x)y(x)=0 \qquad (1.1.14)$$

General Solution of Nonhomogeneous Linear Differential Equation

A **general solution** of Eq.(1.1.13) on some open interval *I* is a solution of the form

$$y(x)=y_h(x)+y_p(x) \qquad (1.1.15)$$

where $y_h(x)=c_1y_1(x)+\cdots+c_ny_n(x)$ is a general solution of the corresponding homogeneous equation (1.1.14) on *I* and $y_p(x)$ is any solution of Eq.(1.1.13) on *I* containing no arbitrary constants.

A **particular solution** of Eq.(1.1.13) on *I* is a solution obtained from Eq.(1.1.15) by assigning specific values to the arbitrary constants $c_1,\cdots,c_n$ in $y_h(x)$.

Initial Value Problem

An **initial value problem** for Eq.(1.1.13) consists of Eq.(1.1.13) and *n* initial conditions

$$y(x_0)=K_0,\ y'(x_0)=K_1,\cdots,y^{(n-1)}(x_0)=K_{n-1} \qquad (1.1.16)$$

where x_0 is some fixed point in the interval *I* considered and $K_0,\cdots,K_{n-1}$ are given numbers.

If the coefficients of Eq.(1.1.13) and $r(x)$ are continuous on some open interval *I* and x_0 is in *I*, then the initial value problem Eq.(1.1.13), Eq.(1.1.16) has a unique solution on *I*.

1.1.4 Method of Undetermined Coefficients

The **method of undetermined coefficients** gives particular solutions $y_p(x)$ of the nonhomogeneous linear differential equation with constant coefficients

$$y^{(n)}(x)+a_{n-1}y^{(n-1)}(x)+\cdots+a_1y'(x)+a_0y(x)=r(x) \qquad (1.1.17)$$

The corresponding homogeneous equation is

$$y^{(n)}(x)+a_{n-1}y^{(n-1)}(x)+\cdots+a_1y'(x)+a_0y(x)=0 \qquad (1.1.18)$$

The rules of the method are as follows.

(1) If $r(x)$ in Eq.(1.1.17) is one of the functions in the first column in Table 1.1.1, choose the corresponding function y_p in the second column.

(2) If a term in your choice for y_p is a solution of the corresponding homogeneous equation Eq.(1.1.18), then multiply y_p by x^k, where k is the smallest positive integer such that no term of $x^k y_p$ is a solution of Eq.(1.1.18).

(3) If $r(x)$ is a sum of functions listed in several lines of Table 1.1.1, first column, then choose for y_p the sum of the functions in the corresponding lines of the second column.

(4) By substituting y_p and its derivatives into Eq.(1.1.17), and comparing the coefficients, determine its undetermined coefficients.

Table 1.1.1 Method of undetermined coefficients

Term in $r(x)$	Choice for y_p
$ke^{\gamma x}$	$Ke^{\gamma x}$
kx^n ($n=0,1,\cdots$)	$K_nx^n+K_{n-1}x^{n-1}+\cdots+K_1x+K_0$
$k\cos\omega x$ $k\sin\omega x$	$K_1\cos\omega x+K_2\sin\omega x$
$ke^{\gamma x}\cos\omega x$ $ke^{\gamma x}\sin\omega x$	$e^{\gamma x}(K_1\cos\omega x+K_2\sin\omega x)$

1.1.5 Method of Variation of Parameters

The **method of variation of parameters** is a method for finding particular solutions $y_p(x)$ of nth order nonhomogeneous linear differential equations

$$y^{(n)}(x)+p_{n-1}(x)y^{(n-1)}(x)+\cdots+p_1(x)y'(x)+p_0(x)y(x)=r(x) \qquad (1.1.19)$$

The method gives a particular solution $y_p(x)$ of Eq.(1.1.19) on I in the form

$$y_p(x)=y_1(x)\int\frac{W_1(x)}{W(x)}r(x)dx+y_2(x)\int\frac{W_2(x)}{W(x)}r(x)dx+\cdots+y_n(x)\int\frac{W_n(x)}{W(x)}r(x)dx \qquad (1.1.20)$$

Hence, $y_1(x),\cdots,y_n(x)$ is a basis of solutions of the corresponding homogeneous equation

$$y^{(n)}(x)+p_{n-1}(x)y^{(n-1)}(x)+\cdots+p_1(x)y'(x)+p_0(x)y(x)=0 \qquad (1.1.21)$$

on I, with Wronskian W, and W_j ($j=1,\cdots,n$) is obtained form W by replacing the jth column of W by the column $[0 \quad 0\cdots 0 \quad 1]$, that is;

$$W=\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix},\ W_1=\begin{vmatrix} 0 & y_2 & \cdots & y_n \\ 0 & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ 0 & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ 1 & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix},\cdots,W_n=\begin{vmatrix} y_1 & y_2 & \cdots & 0 \\ y_1' & y_2' & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & 0 \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & 1 \end{vmatrix} \qquad (1.1.22)$$

This result is obtained by assuming a solution of Eq.(1.1.19) in the form

$$y_p(x)=u_1(x)y_1(x)+\cdots+u_n(x)y_n(x) \qquad (1.1.23)$$

See another book if you want to learn details.

Example-1

Solve the following initial value problem.

$$y'' - 5y' + 6y = x + e^{2x}, \quad y(0) = 1, \quad y'(0) = 0 \tag{1.1.24}$$

Solution

First, we try as the solution of the corresponding homogeneous equation

$$y_h = e^{\lambda x} \tag{1.1.25}$$

Substituting this and its derivatives into the homogeneous equation, we obtain the characteristic equation

$$\lambda^2 - 5\lambda + 6 = 0 \tag{1.1.26}$$

Its roots are

$$\lambda_1 = 2, \quad \lambda_2 = 3 \tag{1.1.27}$$

So that we obtain the general solution of the homogeneous equation

$$y_h = c_1 e^{2x} + c_2 e^{3x} \tag{1.1.28}$$

Next, we get a particular solution by the method of undetermined coefficients.
We choice

$$y_p = K_0 + K_1 x + K_2 x e^{2x} \tag{1.1.29}$$

The first and second terms on the right are chosen for x. The last term on the right, $K_2 x e^{2x}$, is chosen for e^{2x} because e^{2x} is a solution of the corresponding homogeneous equation.
Differentiating this equation, we get

$$y_p' = K_1 + K_2 e^{2x} + 2K_2 x e^{2x}, \quad y_p'' = 4K_2 e^{2x} + 4K_2 x e^{2x} \tag{1.1.30}$$

Substituting these equations into Eq.(1.1.24) and simplifying, we obtain

$$(6K_0 - 5K_1) + 6K_1 x - K_2 e^{2x} = x + e^{2x} \tag{1.1.31}$$

Equating the coefficients of x^0, x and xe^{2x} on both side of this equation, we have

$$6K_0 - 5K_1 = 0, \; 6K_1 = 1, \; K_2 = -1 \tag{1.1.32}$$

These equations yield

$$K_0 = 5/36, \quad K_1 = 1/6, \quad K_2 = -1 \tag{1.1.33}$$

Hence

$$y_p = -xe^{2x} + 5/36 + x/6 \tag{1.1.34}$$

The general solution is

$$y = c_1 e^{2x} + c_2 e^{3x} - xe^{2x} + 5/36 + x/6 \tag{1.1.35}$$

From the initial condition, we get

$$\begin{aligned} y(0) &= c_1 + c_2 + 5/36 = 1 \\ y'(0) &= 2c_1 + 3c_2 - 1 + 1/6 = 0 \end{aligned} \tag{1.1.36}$$

Hence

$$c_1 = 7/4, \quad c_2 = -8/9 \tag{1.1.37}$$

The answer is

$$y = \frac{7}{4} e^{2x} - \frac{8}{9} e^{3x} - xe^{2x} + 5/36 + x/6 \tag{1.1.38}$$

Example-2

Find the general solution of the following differential equation.

$$y'' - 4y' + 4y = 2x^{-3} e^{2x} \tag{1.1.39}$$

Solution

The characteristic equation of the corresponding homogeneous equation is

$$\lambda^2 - 4\lambda + 4 = 0 \tag{1.1.40}$$

Its roots are

$$\lambda_1 = \lambda_2 = 2 \tag{1.1.41}$$

So that we obtain a basis

$$y_1 = e^{2x}, \; y_2 = xe^{2x} \tag{1.1.42}$$

then the general solution of the homogeneous equation is

$$y_h = c_1 e^{2x} + c_2 x e^{2x} \tag{1.1.43}$$

We try to get a particular solution by the method of variation of parameters. The Wronskian and determinants are obtained from Eq.(1.1.22).

$$W = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} = e^{4x}$$

$$W_1 = \begin{vmatrix} 0 & xe^{2x} \\ 1 & e^{2x} + 2xe^{2x} \end{vmatrix} = -xe^{2x}, \quad W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & 1 \end{vmatrix} = e^{2x} \tag{1.1.44}$$

So, we get a particular solution from Eq.(1.1.20)

$$\begin{aligned} y_p &= e^{2x} \int \frac{-xe^{2x}}{e^{4x}} 2x^{-3} e^{2x} dx + xe^{2x} \int \frac{e^{2x}}{e^{4x}} 2x^{-3} e^{2x} dx \\ &= 2x^{-1} e^{2x} - x^{-1} e^{2x} \\ &= x^{-1} e^{2x} \end{aligned} \tag{1.1.45}$$

Hence, the answer is

$$y = c_1 e^{2x} + c_2 xe^{2x} + x^{-1} e^{2x} \tag{1.1.46}$$

Problem-1

Find the general solution of the following differential equations.

(1) $y' - 2y = 5$ (2) $y' + y = \cos x$ (3) $y' - 3y = e^{3x}$

(4) $y'' - 5y = 0$ (5) $y'' - y' - 2y = 0$ (6) $y'' - 10y' + 25y = 0$

(7) $y'' + 10y' + 25 = 0$ (8) $y'' - 6y' + 9y + 12 = 0$ (9) $y'' - 4y' + 3y = e^x$

(10) $y'' + 4y' - 3y = x$ (11) $y'' - 2y' + 2y = e^x \cos x$ (12) $y'' + 9y = \cos 3x + 3x$

(13) $y'' - 2y' + y = x^{1/2} e^x$ (14) $y'' - y' - 2y = (x^{-1/2} + 6x^{1/2}) e^{2x}$ (15) $y''' + 2y'' - y' - 2y = 2e^{2x} + x - 1$

Problem-2

Solve the following initial value problems.

(1) $y'' + 2y' - 3y = 0,\ y(0) = 2, y'(0) = 0$ (2) $y'' - 4y' + 4y = 0,\ y(0) = 1, y'(0) = 0$

(3) $y'' - 2y' + 1 = 0,\ y(0) = 0, y'(0) = 1$ (4) $y'' + 10y' + 25y + 50 = 0,\ y(1) = 2, y'(1) = 1$

(5) $y'' + 3y' + 2y = e^x,\ y(0) = 0, y'(0) = 0$ (6) $y'' + 2y' + 3y = x,\ y(0) = 1, y'(0) = 0$

(7) $y''' - y'' - 4y' + 4y = e^{-x},\ y(0) = 1, y'(0) = 2, y''(0) = 3$

1.2 Systems of Linear Differential Equations

1.2.1 Systems of Linear Differential Equations

Extending the notion of a linear differential equation, a system of n equations in n unknown functions $y_1(x), \cdots, y_n(x)$ of the form

$$\begin{aligned} y_1'(x) &= a_{11}(x)y_1(x) + a_{12}(x)y_2(x) + \cdots + a_{1n}(x)y_n(x) + r_1(x) \\ y_2'(x) &= a_{21}(x)y_1(x) + a_{22}(x)y_2(x) + \cdots + a_{2n}(x)y_n(x) + r_2(x) \\ &\vdots \\ y_n'(x) &= a_{n1}(x)y_1(x) + a_{n2}(x)y_2(x) + \cdots + a_{nn}(x)y_n(x) + r_n(x) \end{aligned} \quad (1.2.1)$$

is called a **linear system**.
In vector form, this becomes

$$\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y} + \boldsymbol{r} \quad (1.2.2)$$

where $\boldsymbol{A} = [a_{ij}] = \begin{bmatrix} a_{11}(x) & \cdots & a_{n1}(x) \\ \vdots & & \vdots \\ a_{n1}(x) & \cdots & a_{nn}(x) \end{bmatrix}, \boldsymbol{y} = \begin{bmatrix} y_1(x) \\ \vdots \\ y_n(x) \end{bmatrix}, \boldsymbol{y}' = \begin{bmatrix} y_1'(x) \\ \vdots \\ y_n'(x) \end{bmatrix}, \boldsymbol{r} = \begin{bmatrix} r_1(x) \\ \vdots \\ r_n(x) \end{bmatrix}$

This system is called **homogeneous** if $\boldsymbol{r} = \boldsymbol{0}$, so that it is

$$\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y} \quad (1.2.3)$$

If $\boldsymbol{r} \neq \boldsymbol{0}$, then Eq.(1.2.2) is called **nonhomogeneous**.

A higher order linear differential equation

$$y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \cdots + p_1(x)y'(x) + p_0(x)y(x) = r(x) \quad (1.2.4)$$

can be written in the form of Eq.(1.2.2), if we set

$$\boldsymbol{y} = \begin{bmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{bmatrix} \quad (1.2.5)$$

hence

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -p_0(x) & -p_1(x) & -p_2(x) & \cdots & -p_{n-1}(x) \end{bmatrix}, \quad \boldsymbol{r} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ r(x) \end{bmatrix} \quad (1.2.6)$$

1.2.2 Homogeneous Linear Systems with Constant Coefficients

In a homogeneous linear system

$$\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y} \quad (1.2.7)$$

we assume that the $n \times n$ matrix $\boldsymbol{A} = [a_{ij}]$ is constant, that is, its entries do not depend on x.

Eigenvalue Problem

To solve Eq.(1.2.7), we try

$$\boldsymbol{y} = \boldsymbol{v}e^{\lambda x} \quad (1.2.8)$$

By substituting this into Eq.(1.2.7) we get

$$\boldsymbol{y}' = \lambda \boldsymbol{v}e^{\lambda x} = \boldsymbol{A}\boldsymbol{y} = \boldsymbol{A}\boldsymbol{v}e^{\lambda x} \quad (1.2.9)$$

Dividing by $e^{\lambda x}$, we are left with the **eigenvalue problem**

$$\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v}, \quad \text{that is,} \quad (\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{v} = \boldsymbol{0} \quad (1.2.10)$$

where λ is an **eigenvalue** of $\boldsymbol{A}$, $\boldsymbol{v}$ is a corresponding **eigenvector** and $\boldsymbol{I}$ is the $n \times n$ unit matrix (identity matrix). For this equation to have a solution $\boldsymbol{v} \neq \boldsymbol{0}$, their coefficient matrix $\boldsymbol{A} - \lambda \boldsymbol{I}$ must be singular, that is,

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = |\boldsymbol{A} - \lambda \boldsymbol{I}| = 0 \quad (1.2.11)$$

This equation is called the **characteristic equation** of $\boldsymbol{A}$. Its solutions are the eigenvalues of $\boldsymbol{A}$.

Now we assume that a characteristic equation is expressed as the form

$$|\boldsymbol{A}-\lambda\boldsymbol{I}|=\prod_{i=1}^{k}(\lambda-\lambda_i)^{m_i}=(\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2}\cdots(\lambda-\lambda_k)^{m_k} \qquad (\sum_{i=1}^{k}m_i=n) \tag{1.2.12}$$

where m_i ($i=1,\cdots,k$) is called the **algebraic multiplicity** of an eigenvalue λ_i which is defined as the multiplicity of the corresponding root of the characteristic equation, and $k_i=n-\mathrm{rank}(\boldsymbol{A}-\lambda_i\boldsymbol{I})$ is called the **geometric multiplicity** of an eigenvalue which is defined as the dimension of the associated eigenspace W_i , $k_i=\dim W_i$, that is, number of linearly independent eigenvectors with that eigenvalue. Both algebraic and geometric multiplicities are integers between (including) 1 and *n*. The algebraic multiplicity m_i and geometric multiplicity k_i may or may not be equal, but we always have $k_i\le m_i$.

(1) If the matrix $\boldsymbol{A}$ in Eq.(1.2.7) has a basis of *n* eigenvectors, that is, a linearly independent set of *n* eigenvectors $\boldsymbol{v}_1,\cdots,\boldsymbol{v}_n$ corresponding to eigenvalues $\lambda_1,\cdots,\lambda_n$, respectively, then the corresponding solutions

$$\boldsymbol{y}_1=\boldsymbol{v}_1e^{\lambda_1 x},\cdots,\boldsymbol{y}_n=\boldsymbol{v}_ne^{\lambda_n x} \tag{1.2.13}$$

form a basis of solution of Eq.(1.2.7), and the corresponding general solution is

$$\boldsymbol{y}=c_1\boldsymbol{y}_1+\cdots+c_n\boldsymbol{y}_n=c_1\boldsymbol{v}_1e^{\lambda_1 x}+\cdots+c_n\boldsymbol{v}_ne^{\lambda_n x} \qquad (c_1,\cdots,c_n \text{ arbitrary}) \tag{1.2.14}$$

It can be expressed in vector form

$$\boldsymbol{y}=[\boldsymbol{y}_1 \ \cdots \ \boldsymbol{y}_n]\begin{bmatrix}c_1\\ \vdots\\ c_n\end{bmatrix}=\boldsymbol{Y}\boldsymbol{c} \tag{1.2.15}$$

This happens when the algebraic multiplicity of all eigenvalue is equal to its geometric multiplicity, that is, $k_i=m_i$ for all λ_i . It is the simplest case that all eigenvalues are different, that is, $k_i=m_i=1$ for all λ_i .

The matrix $\boldsymbol{A}$ is **diagonalizable** by a suitable choice of coordinates if and only if there is a basis of eigenvectors. We choose a nonsingular matrix which entries are eigenvectors

$$\boldsymbol{T}=[\boldsymbol{v}_1 \ \cdots \ \boldsymbol{v}_n] \tag{1.2.16}$$

Taking the similarity transformation by using this matrix, the matrix $\boldsymbol{A}$ becomes a diagonal matrix

$$\boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T}=\mathrm{diag}(\lambda_1,\cdots,\lambda_n)=\begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix} \tag{1.2.17}$$

To apply **diagonalization** to Eq.(1.2.7), we define the new unknown function

$$\boldsymbol{z}=\boldsymbol{T}^{-1}\boldsymbol{y} \quad \text{Then} \quad \boldsymbol{y}=\boldsymbol{T}\boldsymbol{z} \tag{1.2.18}$$

Substituting this in Eq.(1.2.7), we have (note that $\boldsymbol{T}$ is constant)

$$\boldsymbol{T}\boldsymbol{z}'=\boldsymbol{A}\boldsymbol{T}\boldsymbol{z} \tag{1.2.19}$$

We multiply this by $\boldsymbol{T}^{-1}$ from the left, obtaining

$$\boldsymbol{z}'=\boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T}\boldsymbol{z} \tag{1.2.20}$$

Because of Eq.(1.2.17) we can write this in components

$$z_j'=\lambda_j z_j \tag{1.2.21}$$

where $j=1,\cdots,n$. Each equation contains only one of the unknown function $z_1,\cdots,z_n$, and thus can be solved independently of the other equations. We can now solve each of these *n* linear equations, to get

$$z_j(x)=c_je^{\lambda_j x} \tag{1.2.22}$$

These are the components of $\boldsymbol{z}(t)$, and from them we obtain the answer $\boldsymbol{y}=\boldsymbol{T}\boldsymbol{z}$ by Eq.(1.2.18).

(2) If the matrix $\boldsymbol{A}$ in Eq.(1.2.7) has no basis of eigenvectors, we get fewer linearly independent solution of Eq.(1.2.7) and ask how could get a basis of solutions in such a case.

The notion of eigenvector can be generalized to **generalized eigenvectors**. A generalized eigenvector of $\boldsymbol{A}$ is a nonzero vector $\boldsymbol{v}$, which is associated with λ having algebraic multiplicity $k\ge 1$, satisfying

$$(\boldsymbol{A}-\lambda\boldsymbol{I})^k\boldsymbol{v}=\boldsymbol{0} \quad \text{and} \quad (\boldsymbol{A}-\lambda\boldsymbol{I})^{k-1}\boldsymbol{v}\ne\boldsymbol{0} \tag{1.2.23}$$

The set of all generalized eigenvectors for a given λ forms the generalized eigenspace for λ . That is,

$$\boldsymbol{v},\ (\boldsymbol{A}-\lambda\boldsymbol{I})\boldsymbol{v},\ (\boldsymbol{A}-\lambda\boldsymbol{I})^2\boldsymbol{v},\cdots,\ (\boldsymbol{A}-\lambda\boldsymbol{I})^{k-1}\boldsymbol{v} \tag{1.2.24}$$

are linearly independent. Ordinary eigenvectors and eigenspaces are obtained for $k=1$.

The example of how to obtain the generalized eigenvectors for a given λ is as follows.
First, we obtain an eigenvector $\boldsymbol{v}_1$ from $(\boldsymbol{A}-\lambda\boldsymbol{I})\boldsymbol{v}_1=\boldsymbol{0}$.
Next, we obtain a generalized eigenvector $\boldsymbol{v}_2$ from $(\boldsymbol{A}-\lambda\boldsymbol{I})\boldsymbol{v}_2=\boldsymbol{v}_1$. Hence $(\boldsymbol{A}-\lambda\boldsymbol{I})^2\boldsymbol{v}_2=(\boldsymbol{A}-\lambda\boldsymbol{I})\boldsymbol{v}_1=\boldsymbol{0}$.
…
Finally, we obtain a generalized eigenvector $\boldsymbol{v}_k$ from $(\boldsymbol{A}-\lambda\boldsymbol{I})\boldsymbol{v}_k=\boldsymbol{v}_{k-1}$. Hence $(\boldsymbol{A}-\lambda\boldsymbol{I})^k\boldsymbol{v}_k=\boldsymbol{0}$.

A solution for a generalized eigenvector $\boldsymbol{v}_k$ corresponding to an eigenvalue λ is

$$\begin{aligned}\boldsymbol{y}_k&=e^{\lambda x}\left[\boldsymbol{v}_k+x(\boldsymbol{A}-\lambda\boldsymbol{I})\boldsymbol{v}_k+\frac{x^2}{2!}(\boldsymbol{A}-\lambda\boldsymbol{I})^2\boldsymbol{v}_k+\cdots+\frac{x^{k-1}}{(k-1)!}(\boldsymbol{A}-\lambda\boldsymbol{I})^{k-1}\boldsymbol{v}_k\right]\\&=e^{\lambda x}\left[\boldsymbol{v}_k+x\boldsymbol{v}_{k-1}+\frac{x^2}{2!}\boldsymbol{v}_{k-2}+\cdots+\frac{x^{k-1}}{(k-1)!}\boldsymbol{v}_1\right]\end{aligned}\tag{1.2.25}$$

If a matrix $\boldsymbol{A}$ has no basis of eigenvectors, then matrix is not diagonalizable and is transformed to **Jordan normal form** (**Jordan canonical form**) using eigenvectors and generalized eigenvectors such as

$$\boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T}=\boldsymbol{J}=\begin{bmatrix}\boldsymbol{J}_1&\boldsymbol{0}&\cdots&\boldsymbol{0}\\\boldsymbol{0}&\boldsymbol{J}_2&&\vdots\\\vdots&&\ddots&\boldsymbol{0}\\\boldsymbol{0}&\cdots&\boldsymbol{0}&\boldsymbol{J}_r\end{bmatrix},\quad \boldsymbol{J}_i=\begin{bmatrix}\lambda_i&1&&0\\0&\lambda_i&\ddots&\\\vdots&\ddots&\ddots&1\\0&\cdots&0&\lambda_i\end{bmatrix}\tag{1.2.26}$$

where each $\boldsymbol{J}_i$ is called a **Jordan block**. See the book on the linear algebra in detail.

If $\boldsymbol{A}$ is a real matrix, its Jordan form can still be non-real, however there exists a real invertible matrix $\boldsymbol{T}$ such that $\boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T}=\boldsymbol{J}$ is a real block diagonal matrix with each block being a real Jordan block. A real Jordan block is either identical to a complex Jordan block (if the corresponding eigenvalue λ_i is real), or is a block matrix itself, consisting of 2×2 blocks as follows (for non-real eigenvalue $\lambda_i=\sigma_i+i\omega_i$). The diagonal blocks are identical, of the form

$$\boldsymbol{C}_i=\begin{bmatrix}\sigma_i&\omega_i\\-\omega_i&\sigma_i\end{bmatrix}\tag{1.2.27}$$

and describe multiplication by λ_i in the complex plane. The superdiagonal blocks are 2×2 identity matrices. The full **real Jordan block** is given by

$$\boldsymbol{J}_i=\begin{bmatrix}\boldsymbol{C}_i&\boldsymbol{I}&\cdots&\boldsymbol{0}\\\boldsymbol{0}&\boldsymbol{C}_i&\ddots&\vdots\\\vdots&&\ddots&\boldsymbol{I}\\\boldsymbol{0}&\cdots&\boldsymbol{0}&\boldsymbol{C}_i\end{bmatrix}\tag{1.2.28}$$

This real Jordan form is a consequence of the complex Jordan form. For a real matrix the non-real eigenvectors and generalized eigenvectors can always be chosen to form complex conjugate pairs. Taking the real and imaginary part (linear combination of the vector and its conjugate), the matrix has this form in the new basis. For example, if $\boldsymbol{A}$ is a 2×2 real matrix with eigenvalues $\lambda=\sigma\pm i\omega$ and eigenvectors $\boldsymbol{v}=\boldsymbol{p}\pm i\boldsymbol{q}$, then we can choose $\boldsymbol{T}=[\boldsymbol{p}\quad\boldsymbol{q}]$. Hence

$$\boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T}=\begin{bmatrix}\sigma&\omega\\-\omega&\sigma\end{bmatrix}\tag{1.2.29}$$

1.2.3 Nonhomogeneous Linear Systems

A general solution of a nonhomogeneous linear system

$$\boldsymbol{y}'=\boldsymbol{A}\boldsymbol{y}+\boldsymbol{r}\tag{1.2.30}$$

with the vector $\boldsymbol{r}$ not identically zero is a solution of the form

$$\boldsymbol{y}=\boldsymbol{y}_h+\boldsymbol{y}_p\tag{1.2.31}$$

where $\boldsymbol{y}_h$ is a general solution of the corresponding homogeneous linear system and $\boldsymbol{y}_p$ is a particular solution of Eq.(1.2.30) containing no arbitrary constants.

A general solution of the corresponding homogeneous linear system is

$$\boldsymbol{y}_h = [\boldsymbol{y}_1 \ \cdots \ \boldsymbol{y}_n]\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \boldsymbol{Y}\boldsymbol{c} \tag{1.2.32}$$

where $\boldsymbol{Y} = [\boldsymbol{y}_1 \ \cdots \ \boldsymbol{y}_n]$, $\boldsymbol{y}_1, \cdots, \boldsymbol{y}_n$ is a basis of the corresponding homogeneous linear system, and $\boldsymbol{c}$ is constant.

Similar to higher order linear differential equations, if all entries of $\boldsymbol{r}$ are contained in the first column in Table 1.1.1, then the method of undetermined coefficients can be used to obtain $\boldsymbol{y}_p$.

The method of variation of parameters gives a particular solution $\boldsymbol{y}_p$ in the form

$$\boldsymbol{y}_p = \boldsymbol{Y}\int \boldsymbol{Y}^{-1}\boldsymbol{r}\, dx \tag{1.2.33}$$

Matrix Exponential

Let $\boldsymbol{X}$ be an $n \times n$ matrix $\boldsymbol{X} = [x_{ij}]$, the **matrix exponential** is defined as

$$e^{\boldsymbol{X}} = \sum_{k=0}^{\infty} \frac{1}{k!}\boldsymbol{X}^k = \boldsymbol{I} + \boldsymbol{X} + \frac{1}{2!}\boldsymbol{X}^2 + \cdots + \frac{1}{k!}\boldsymbol{X}^k + \cdots \tag{1.2.34}$$

which is similar to the Taylor series of an exponential function

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!}x^k = 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{k!}x^k + \cdots \tag{1.2.35}$$

Some properties of the matrix exponential are shown in Table 3.2.1.

Note the following relation, in general

$$e^{\boldsymbol{X}} \neq \begin{bmatrix} e^{x_{11}} & \cdots & e^{x_{1n}} \\ \vdots & & \vdots \\ e^{x_{n1}} & \cdots & e^{x_{nn}} \end{bmatrix}$$

Let $\boldsymbol{X} = \boldsymbol{A}x$ in Eq.(1.2.34), that is,

$$e^{\boldsymbol{A}x} = \sum_{k=0}^{\infty} \frac{1}{k!}\boldsymbol{A}^k x^k = \boldsymbol{I} + \boldsymbol{A}x + \frac{1}{2!}\boldsymbol{A}^2 x^2 + \cdots + \frac{1}{k!}\boldsymbol{A}^k x^k + \cdots \tag{1.2.36}$$

Differentiating this matrix exponential, we obtain

$$\frac{de^{\boldsymbol{A}x}}{dx} = \boldsymbol{A} + \frac{2}{2!}\boldsymbol{A}^2 x + \cdots + \frac{k}{k!}\boldsymbol{A}^k x^{k-1} + \cdots = \boldsymbol{A}\left(\boldsymbol{I} + \boldsymbol{A}x + \cdots + \frac{1}{(k-1)!}\boldsymbol{A}^{k-1}x^{k-1} + \cdots\right) = \boldsymbol{A}e^{\boldsymbol{A}x} \tag{1.2.37}$$

Hence $e^{\boldsymbol{A}x}$ is a solution of the homogeneous linear system $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$.

Using this matrix exponential, the solution of the initial value problem for the homogeneous linear system

$$\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}, \quad \boldsymbol{y}(0) = \boldsymbol{y}_0 \tag{1.2.38}$$

is expressed as

$$\boldsymbol{y} = e^{\boldsymbol{A}x}\boldsymbol{y}_0 \tag{1.2.39}$$

Furthermore, the solution of the initial value problem for the nonhomogeneous linear system

$$\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y} + \boldsymbol{r}, \quad \boldsymbol{y}(0) = \boldsymbol{y}_0 \tag{1.2.40}$$

is expressed as

$$\boldsymbol{y} = e^{\boldsymbol{A}x}\boldsymbol{y}_0 + \int_0^x e^{\boldsymbol{A}(x-\tau)}\boldsymbol{r}(\tau)d\tau \tag{1.2.41}$$

Example-1

Solve the following initial value problem.

$$\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y} + \boldsymbol{r} = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}\boldsymbol{y} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}x, \quad \boldsymbol{y}(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \tag{1.2.42}$$

Solution

The characteristic equation of the corresponding homogeneous linear system is

$$|\lambda\boldsymbol{I} - \boldsymbol{A}| = \begin{vmatrix} \lambda - 1 & 2 \\ -3 & \lambda + 4 \end{vmatrix} = \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) = 0 \tag{1.2.43}$$

This gives the eigenvalues

$$\lambda_1 = -1, \quad \lambda_2 = -2 \tag{1.2.44}$$

Then eigenvectors are obtained from the equation

$$(\lambda \boldsymbol{I}-\boldsymbol{A})\boldsymbol{v}=\begin{bmatrix}\lambda-1 & 2\\ -3 & \lambda+4\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix} \tag{1.2.45}$$

For $\lambda_1=-1$, this gives

$$v_1=v_2 \tag{1.2.46}$$

So, we can take

$$\boldsymbol{v}_1=\begin{bmatrix}1\\ 1\end{bmatrix}. \qquad \text{Then } \boldsymbol{y}_1=\boldsymbol{v}_1 e^{\lambda_1 x}=\begin{bmatrix}1\\ 1\end{bmatrix}e^{-x} \tag{1.2.47}$$

For $\lambda_2=-2$,

$$3v_1=2v_2 \tag{1.2.48}$$

We can take

$$\boldsymbol{v}_2=\begin{bmatrix}2\\ 3\end{bmatrix}. \qquad \text{Then } \boldsymbol{y}_2=\boldsymbol{v}_2 e^{\lambda_2 x}=\begin{bmatrix}2\\ 3\end{bmatrix}e^{-2x} \tag{1.2.49}$$

Thus, we obtain the general solution of the homogeneous linear system

$$\boldsymbol{y}_h=c_1\begin{bmatrix}1\\ 1\end{bmatrix}e^{-x}+c_2\begin{bmatrix}2\\ 3\end{bmatrix}e^{-2x} \tag{1.2.50}$$

From the form of Eq.(1.2.42), a particular solution is assumed in the form

$$\boldsymbol{y}_p=\boldsymbol{u}_0+\boldsymbol{u}_1 x \tag{1.2.51}$$

to determine the vectors $\boldsymbol{u}_0$ and $\boldsymbol{u}_1$.
Substituting this and its derivative into Eq.(1.2.42), we obtain

$$\begin{bmatrix}u_{11}\\ u_{12}\end{bmatrix}=\begin{bmatrix}1 & -2\\ 3 & -4\end{bmatrix}\left(\begin{bmatrix}u_{01}\\ u_{02}\end{bmatrix}+\begin{bmatrix}u_{11}\\ u_{12}\end{bmatrix}x\right)+\begin{bmatrix}1\\ 2\end{bmatrix}x \tag{1.2.52}$$

Equating the coefficients of x^0 and x on both side of this equation, we get

$$\begin{bmatrix}u_{01}\\ u_{02}\end{bmatrix}=\begin{bmatrix}1/2\\ 1/4\end{bmatrix}, \quad \begin{bmatrix}u_{11}\\ u_{12}\end{bmatrix}=\begin{bmatrix}0\\ 1/2\end{bmatrix} \tag{1.2.53}$$

Hence, a particular solution is

$$\boldsymbol{y}_p=\begin{bmatrix}1/2\\ 1/4+x/2\end{bmatrix} \tag{1.2.54}$$

From this, we obtain the general solution of the nonhomogeneous linear system

$$\boldsymbol{y}=c_1\begin{bmatrix}1\\ 1\end{bmatrix}e^{-x}+c_2\begin{bmatrix}2\\ 3\end{bmatrix}e^{-2x}+\begin{bmatrix}1/2\\ 1/4+x/2\end{bmatrix} \tag{1.2.55}$$

Finally, from the initial condition, we obtain

$$\boldsymbol{y}(0)=c_1\begin{bmatrix}1\\ 1\end{bmatrix}+c_2\begin{bmatrix}2\\ 3\end{bmatrix}+\begin{bmatrix}1/2\\ 1/4\end{bmatrix}=\begin{bmatrix}2\\ 2\end{bmatrix} \tag{1.2.56}$$

Hence

$$c_1=1, \quad c_2=1/4 \tag{1.2.57}$$

The answer is

$$\boldsymbol{y}=\begin{bmatrix}1\\ 1\end{bmatrix}e^{-x}+\frac{1}{4}\begin{bmatrix}2\\ 3\end{bmatrix}e^{-2x}+\begin{bmatrix}1/2\\ 1/4+x/2\end{bmatrix} \tag{1.2.58}$$

Next, we find a particular solution by the method of variation of parameters.
Using the basis of the homogeneous linear system, we have

$$\boldsymbol{Y}=\begin{bmatrix}e^{-x} & 2e^{-2x}\\ e^{-x} & 3e^{-2x}\end{bmatrix} \tag{1.2.59}$$

Hence, we get a particular solution

$$\boldsymbol{y}_p = \boldsymbol{Y}\int \boldsymbol{Y}^{-1}\boldsymbol{r}\,dx = \begin{bmatrix} e^{-x} & 2e^{-2x} \\ e^{-x} & 3e^{-2x} \end{bmatrix} \int e^{3x} \begin{bmatrix} 3e^{-2x} & -2e^{-2x} \\ -e^{-x} & e^{-x} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} x dx = \begin{bmatrix} e^{-x} & 2e^{-2x} \\ e^{-x} & 3e^{-2x} \end{bmatrix} \int \begin{bmatrix} -xe^{x} \\ xe^{2x} \end{bmatrix} dx$$
$$= \begin{bmatrix} e^{-x} & 2e^{-2x} \\ e^{-x} & 3e^{-2x} \end{bmatrix} \begin{bmatrix} (1-x)e^{x} \\ (x/2-1/4)e^{2x} \end{bmatrix} = \begin{bmatrix} 1/2 \\ x/2+1/4 \end{bmatrix} \tag{1.2.60}$$

This agrees with Eq.(1.2.55).

Now, we try to solve Eq.(1.2.42) by the method of diagonalization.
From the eigenvectors of this system, we get the similarity transformation matrix

$$\boldsymbol{T} = [\boldsymbol{v}_1 \quad \boldsymbol{v}_2] = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}. \qquad \text{Hence} \quad \boldsymbol{T}^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \tag{1.2.61}$$

Using this matrix, we define the new unknown function

$$\boldsymbol{z} = \boldsymbol{T}^{-1}\boldsymbol{y} \qquad \text{Then} \qquad \boldsymbol{y} = \boldsymbol{T}\boldsymbol{z} \tag{1.2.62}$$

Substituting this into Eq.(1.2.42), we obtain the diagonalized system

$$\boldsymbol{z}' = \boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T}\boldsymbol{z} + \boldsymbol{T}^{-1}\boldsymbol{r} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \boldsymbol{z} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} x \tag{1.2.63}$$

thus

$$\begin{aligned} z_1' &= -z_1 - x \\ z_2' &= -2z_2 + x \end{aligned} \tag{1.2.64}$$

Solving each of these equations, we obtain

$$\begin{aligned} z_1 &= c_1 e^{-x} + 1 - x \\ z_2 &= c_2 e^{-2x} - 1/4 + x/2 \end{aligned} \tag{1.2.65}$$

From this and the similarity transformation matrix, we get the general solution of the nonhomogeneous linear system

$$\boldsymbol{y} = \boldsymbol{T}\boldsymbol{z} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 e^{-x} + 1 - x \\ c_2 e^{-2x} - 1/4 + x/2 \end{bmatrix} = \begin{bmatrix} c_1 e^{-x} + 2c_1 e^{-2x} + 1/2 \\ c_1 e^{-x} + 3c_1 e^{-2x} + 1/4 + x/2 \end{bmatrix} \tag{1.2.66}$$

This is identical with Eq.(1.2.55).

Example-2

Find the general solution of the following linear system.

$$\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y} = \begin{bmatrix} 0 & 1 \\ -25 & -6 \end{bmatrix} \boldsymbol{y} \tag{1.2.67}$$

Solution

The characteristic equation of this equation is

$$|\lambda \boldsymbol{I} - \boldsymbol{A}| = \begin{vmatrix} \lambda & -1 \\ 25 & \lambda+6 \end{vmatrix} = \lambda^2 + 6\lambda + 25 = 0 \tag{1.2.68}$$

This gives the eigenvalues

$$\lambda_1 = -3 + 4i, \quad \lambda_2 = -3 - 4i \tag{1.2.69}$$

Then eigenvectors are obtained from

$$(\lambda \boldsymbol{I} - \boldsymbol{A})\boldsymbol{v} = \begin{bmatrix} \lambda & -1 \\ 25 & \lambda+6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{1.2.70}$$

For $\lambda_1 = -3 + 4i$, this becomes

$$(-3+4i)v_1 = v_2 \tag{1.2.71}$$

So we can choose

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -3+4i \end{bmatrix} \tag{1.2.72}$$

For $\lambda_2 = -3-4i$, we find

$$\boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -3-4i \end{bmatrix} \tag{1.2.73}$$

This gives the general solution

$$\boldsymbol{y} = c_1 \begin{bmatrix} 1 \\ -3+4i \end{bmatrix} e^{(-3+4i)x} + c_2 \begin{bmatrix} 1 \\ -3-4i \end{bmatrix} e^{(-3-4i)x} \tag{1.2.74}$$

This solution is complex, but we can obtain a real general solution.
Using $e^{(\alpha \pm i\beta)x} = e^{\alpha x}(\cos\beta t \pm \sin\beta t)$ and collecting the real and imaginary parts, we obtain

$$\begin{bmatrix} 1 \\ -3+4i \end{bmatrix} e^{(-3+4i)x} = \begin{bmatrix} e^{-3x}(\cos 4x + i\sin 4x) \\ (-3+4i)e^{-3x}(\cos 4x + i\sin 4x) \end{bmatrix}$$
$$= \begin{bmatrix} e^{-3x}\cos 4x \\ e^{-3x}(-3\cos 4x - 4\sin 4x) \end{bmatrix} + i \begin{bmatrix} e^{-3x}\sin 4x \\ e^{-3x}(-3\sin 4x + 4\cos 4x) \end{bmatrix}$$

and (1.2.75)

$$\begin{bmatrix} 1 \\ -3-4i \end{bmatrix} e^{(-3-4i)x} = \begin{bmatrix} e^{-3x}(\cos 4x - i\sin 4x) \\ (-3-4i)e^{-3x}(\cos 4x - i\sin 4x) \end{bmatrix}$$
$$= \begin{bmatrix} e^{-3x}\cos 4x \\ e^{-3x}(-3\cos 4x - 4\sin 4x) \end{bmatrix} - i \begin{bmatrix} e^{-3x}\sin 4x \\ e^{-3x}(-3\sin 4x + 4\cos 4x) \end{bmatrix}$$

The real and imaginary parts on the right side of this equation are real solutions of Eq.(1.2.67). They form a basis because their Wronskian is not zero.

Hence the real general solution is

$$\boldsymbol{y} = \tilde{c}_1 \begin{bmatrix} e^{-3x}\cos 4x \\ e^{-3x}(-3\cos 4x - 4\sin 4x) \end{bmatrix} + \tilde{c}_2 \begin{bmatrix} e^{-3x}\sin 4x \\ e^{-3x}(-3\sin 4x + 4\cos 4x) \end{bmatrix} \tag{1.2.76}$$

Now, we get the real general solution by another method.
We obtain the transformation matrix by using the real and imaginary parts of these eigenvectors,

$$\boldsymbol{T} = \begin{bmatrix} 1 & 0 \\ -3 & 4 \end{bmatrix} \tag{1.2.77}$$

Hence, we define the new unknown function

$$\boldsymbol{z} = \boldsymbol{T}^{-1}\boldsymbol{y} \qquad \text{Then} \qquad \boldsymbol{y} = \boldsymbol{T}\boldsymbol{z} \tag{1.2.78}$$

Substituting this into Eq.(1.2.67), we obtain the equation

$$\boldsymbol{z}' = \boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T}\boldsymbol{z} = \begin{bmatrix} -3 & 4 \\ -4 & -3 \end{bmatrix} \boldsymbol{z} \tag{1.2.79}$$

The real general solution of this equation is

$$\boldsymbol{z} = e^{-3x} \begin{bmatrix} \cos 4x & \sin 4x \\ -\sin 4x & \cos 4x \end{bmatrix} \begin{bmatrix} \tilde{c}_1 \\ \tilde{c}_2 \end{bmatrix} = \tilde{c}_1 \begin{bmatrix} e^{-3x}\cos 4x \\ -e^{-3x}\sin 4x \end{bmatrix} + \tilde{c}_2 \begin{bmatrix} e^{-3x}\sin 4x \\ e^{-3x}\cos 4x \end{bmatrix} \tag{1.2.80}$$

Hence the answer is

$$\boldsymbol{y} = \boldsymbol{T}\boldsymbol{z} = \begin{bmatrix} 1 & 0 \\ -3 & 4 \end{bmatrix} \left\{ \tilde{c}_1 \begin{bmatrix} e^{-3x}\cos 4x \\ -e^{-3x}\sin 4x \end{bmatrix} + \tilde{c}_2 \begin{bmatrix} e^{-3x}\sin 4x \\ e^{-3x}\cos 4x \end{bmatrix} \right\}$$
$$= \tilde{c}_1 \begin{bmatrix} e^{-3x}\cos 4x \\ e^{-3x}(-3\cos 4x - 4\sin 4x) \end{bmatrix} + \tilde{c}_2 \begin{bmatrix} e^{-3x}\sin 4x \\ e^{-3x}(-3\sin 4x + 4\cos 4x) \end{bmatrix} \tag{1.2.81}$$

This agrees with Eq.(1.2.76).

Example-3

Find the general solutions of the following linear systems.

$$\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$$

(1) $\boldsymbol{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (2) $\boldsymbol{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ (3) $\boldsymbol{A} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ (4) $\boldsymbol{A} = \begin{bmatrix} 1 & -4 & -3 \\ 0 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

Solution

(1) The matrix $\boldsymbol{A}$ is diagonal. The characteristic equation is $|\lambda \boldsymbol{I} - \boldsymbol{A}| = (\lambda - 2)^3 = 0$. This gives the eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = 2$. Thus the algebraic multiplicity $m = 3$ and the geometric multiplicity $k = 3$ for the eigenvalue $\lambda = 2$. Hence we obtain three eigenvectors and a basis of the system such as

$$\boldsymbol{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2x},\ \boldsymbol{y}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{2x},\ \boldsymbol{y}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2x} \tag{1.2.82}$$

Hence we obtain a general solution of the system

$$\boldsymbol{y} = c_1\boldsymbol{y}_1 + c_2\boldsymbol{y}_2 + c_3\boldsymbol{y}_3 = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2x} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{2x} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2x} = \begin{bmatrix} c_1 e^{2x} \\ c_2 e^{2x} \\ c_3 e^{2x} \end{bmatrix} \tag{1.2.83}$$

(2) The matrix $\boldsymbol{A}$ is a Jordan normal form. The characteristic equation is $|\lambda \boldsymbol{I} - \boldsymbol{A}| = (\lambda - 2)^3 = 0$ which is the same characteristic equation as the previous example (1), but the algebraic multiplicity $m = 3$ and the geometric multiplicity $k = 2$ for the eigenvalue $\lambda = 2$. Hence we obtain two eigenvectors and a generalized eigenvector, and a basis of the system, for example

$$\boldsymbol{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2x},\ \boldsymbol{y}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{2x},\ \boldsymbol{y}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2x} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} x e^{2x} \tag{1.2.84}$$

Hence we obtain a general solution of the system

$$\boldsymbol{y} = c_1\boldsymbol{y}_1 + c_2\boldsymbol{y}_2 + c_3\boldsymbol{y}_3 = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2x} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{2x} + c_3 \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2x} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} x e^{2x} \right) = \begin{bmatrix} c_1 e^{2x} \\ c_2 e^{2x} + c_3 x e^{2x} \\ c_3 e^{2x} \end{bmatrix} \tag{1.2.85}$$

(3) The matrix $\boldsymbol{A}$ is a Jordan normal form. The characteristic equation is $|\lambda \boldsymbol{I} - \boldsymbol{A}| = (\lambda - 2)^3 = 0$, but the algebraic multiplicity $m = 3$ and the geometric multiplicity $k = 1$ for the eigenvalue $\lambda = 2$. Hence we obtain an eigenvector and two generalized eigenvectors, and a basis of the system, for example

$$\boldsymbol{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2x},\ \boldsymbol{y}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{2x} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} x e^{2x},\ \boldsymbol{y}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2x} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} x e^{2x} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{x^2}{2} e^{2x} \tag{1.2.86}$$

Hence we obtain a general solution of the system

$$\begin{aligned} \boldsymbol{y} &= c_1\boldsymbol{y}_1 + c_2\boldsymbol{y}_2 + c_3\boldsymbol{y}_3 = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2x} + c_2 \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{2x} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} x e^{2x} \right) + c_3 \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2x} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} x e^{2x} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{x^2}{2} e^{2x} \right) \\ &= \begin{bmatrix} c_1 e^{2x} + c_2 x e^{2x} + \frac{1}{2} c_3 x^2 e^{2x} \\ c_2 e^{2x} + c_3 x e^{2x} \\ c_3 e^{2x} \end{bmatrix} \end{aligned} \tag{1.2.87}$$

(4) The characteristic equation is $|\lambda \boldsymbol{I}-\boldsymbol{A}|=(\lambda-3)(\lambda-2)^2=0$. This gives the eigenvalues $\lambda_1=3,\ \lambda_2=\lambda_3=2$. For the eigenvalue $\lambda_1=3$, the algebraic multiplicity $m_1=1$ and the geometric multiplicity $k_1=1$. Then an eigenvector for λ_1 is obtained from the equation

$$(\lambda_1\boldsymbol{I}-\boldsymbol{A})\boldsymbol{v}_1=\begin{bmatrix}-2 & -4 & -3\\ 0 & 0 & 1\\ 1 & 2 & 0\end{bmatrix}\begin{bmatrix}v_{11}\\ v_{12}\\ v_{13}\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \tag{1.2.88}$$

Hence
$$\begin{cases}v_{13}=0\\ v_{11}=-2v_{12}\end{cases} \tag{1.2.89}$$

So we can take

$$\boldsymbol{v}_1=\begin{bmatrix}v_{11}\\ v_{12}\\ v_{13}\end{bmatrix}=\begin{bmatrix}-2\\1\\0\end{bmatrix} \tag{1.2.90}$$

Then

$$\boldsymbol{y}_1=\boldsymbol{v}_1e^{\lambda_1x}=\begin{bmatrix}-2\\1\\0\end{bmatrix}e^{3x} \tag{1.2.91}$$

For the eigenvalue $\lambda_2=2$, the algebraic multiplicity $m_2=2$ and the geometric multiplicity $k_2=1$. Then an eigenvector for λ_2 is obtained from the equation

$$(\lambda_2\boldsymbol{I}-\boldsymbol{A})\boldsymbol{v}_2=\begin{bmatrix}-1 & -4 & -3\\ 0 & 1 & 1\\ 1 & 2 & 1\end{bmatrix}\begin{bmatrix}v_{21}\\ v_{22}\\ v_{23}\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \tag{1.2.92}$$

Hence

$$\begin{cases}v_{23}=-v_{22}\\ v_{21}=-v_{22}\end{cases} \tag{1.2.93}$$

So we can take

$$\boldsymbol{v}_2=\begin{bmatrix}v_{21}\\ v_{22}\\ v_{23}\end{bmatrix}=\begin{bmatrix}1\\-1\\1\end{bmatrix} \tag{1.2.94}$$

Then

$$\boldsymbol{y}_2=\boldsymbol{v}_2e^{\lambda_2x}=\begin{bmatrix}1\\-1\\1\end{bmatrix}e^{2x} \tag{1.2.95}$$

Hence a generalized eigenvector is obtained from the equation

$$(\lambda_2\boldsymbol{I}-\boldsymbol{A})\boldsymbol{v}_3=\begin{bmatrix}-1 & -4 & -3\\ 0 & 1 & 1\\ 1 & 2 & 1\end{bmatrix}\begin{bmatrix}v_{31}\\ v_{32}\\ v_{33}\end{bmatrix}=\begin{bmatrix}1\\-1\\1\end{bmatrix} \tag{1.2.96}$$

Hence

$$\begin{cases}v_{32}+v_{33}=-1\\ v_{31}+v_{32}=2\end{cases} \tag{1.2.97}$$

We can take, for example,

$$\boldsymbol{v}_3=\begin{bmatrix}v_{31}\\ v_{32}\\ v_{33}\end{bmatrix}=\begin{bmatrix}2\\0\\-1\end{bmatrix} \tag{1.2.98}$$

Then

$$\boldsymbol{y}_3=\begin{bmatrix}2\\0\\-1\end{bmatrix}e^{2x}+\begin{bmatrix}1\\-1\\1\end{bmatrix}xe^{2x} \tag{1.2.99}$$

Thus, we obtain a general solution of the linear system

$$\boldsymbol{y} = c_1\boldsymbol{y}_1 + c_2\boldsymbol{y}_2 + c_3\boldsymbol{y}_3 = c_1\begin{bmatrix}-2\\1\\0\end{bmatrix}e^{3x} + c_2\begin{bmatrix}1\\-1\\1\end{bmatrix}e^{2x} + c_3\left(\begin{bmatrix}2\\0\\-1\end{bmatrix}e^{2x} + \begin{bmatrix}1\\-1\\1\end{bmatrix}xe^{2x}\right) \quad (1.2.100)$$

From the eigenvectors and the generalized eigenvector, we get the similarity transformation matrix

$$\boldsymbol{T} = \begin{bmatrix}-2 & 1 & 2\\1 & -1 & 0\\0 & 1 & -1\end{bmatrix} \quad (1.2.101)$$

Hence the matrix $\boldsymbol{A}$ is transformed to a Jordan normal form by this $\boldsymbol{T}$

$$\boldsymbol{T}^{-1}\boldsymbol{AT} = \begin{bmatrix}3 & 0 & 0\\0 & 2 & 1\\0 & 0 & 2\end{bmatrix} \quad (1.2.102)$$

We can choose another generalized eigenvector as long as Eq.(1.2.97) is satisfied, for example,

$$\boldsymbol{v}_3 = \begin{bmatrix}v_{31}\\v_{32}\\v_{33}\end{bmatrix} = \begin{bmatrix}3\\-1\\0\end{bmatrix} \quad (1.2.103)$$

In this case, we have

$$\boldsymbol{y}_3 = \begin{bmatrix}3\\-1\\0\end{bmatrix}e^{2x} + \begin{bmatrix}1\\-1\\1\end{bmatrix}xe^{2x} \quad (1.2.104)$$

The matrix $\boldsymbol{A}$ is transformed to the same Jordan normal form even if this chosen generalized eigenvector is used. Both $\boldsymbol{y}_3$ in Eq.(1.2.99) and Eq.(1.2.104) are the solutions of the linear system though they seem to be different.

Problem

Solve the following initial value problems.

$$\boldsymbol{y}' = \boldsymbol{Ay}, \quad \boldsymbol{y}(0) = \boldsymbol{y}_0$$

(1) $\boldsymbol{A} = \begin{bmatrix}7 & 5\\1 & 3\end{bmatrix}, \boldsymbol{y}_0 = \begin{bmatrix}1\\1\end{bmatrix}$ (2) $\boldsymbol{A} = \begin{bmatrix}-1 & 2\\-2 & -5\end{bmatrix}, \boldsymbol{y}_0 = \begin{bmatrix}1\\1\end{bmatrix}$ (3) $\boldsymbol{A} = \begin{bmatrix}1 & 5\\-2 & -1\end{bmatrix}, \boldsymbol{y}_0 = \begin{bmatrix}5\\-1\end{bmatrix}$

(4) $\boldsymbol{A} = \begin{bmatrix}1 & -2\\4 & -3\end{bmatrix}, \boldsymbol{y}_0 = \begin{bmatrix}0\\1\end{bmatrix}$ (5) $\boldsymbol{A} = \begin{bmatrix}-2 & 0\\0 & -2\end{bmatrix}, \boldsymbol{y}_0 = \begin{bmatrix}1\\-1\end{bmatrix}$ (6) $\boldsymbol{A} = \begin{bmatrix}0 & 1 & 0\\0 & -1 & 2\\0 & -2 & -6\end{bmatrix}, \boldsymbol{y}_0 = \begin{bmatrix}1\\0\\0\end{bmatrix}$

(7) $\boldsymbol{A} = \begin{bmatrix}0 & 1 & -3\\2 & -1 & -3\\2 & 1 & -5\end{bmatrix}, \boldsymbol{y}_0 = \begin{bmatrix}y_{10}\\y_{20}\\y_{30}\end{bmatrix}$ (8) $\boldsymbol{A} = \begin{bmatrix}2 & 6 & 4\\-4 & -8 & -4\\2 & 3 & 0\end{bmatrix}, \boldsymbol{y}_0 = \begin{bmatrix}y_{10}\\y_{20}\\y_{30}\end{bmatrix}$ (9) $\boldsymbol{A} = \begin{bmatrix}2 & 6 & 4\\-3 & -6 & -2\\1 & 1 & -2\end{bmatrix}, \boldsymbol{y}_0 = \begin{bmatrix}y_{10}\\y_{20}\\y_{30}\end{bmatrix}$

1.3 Series Solutions of Differential Equations

The power series method gives solutions of differential equations in the form of power series.

1.3.1 Power Series

A **power series** is an infinite series of the form

$$\sum_{m=0}^{\infty} a_m (x-x_0)^m = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots \tag{1.3.1}$$

We assume the variable x, the **center** x_0 and the **coefficients** $a_0, a_1, \cdots$ to be real.
The ***n*th partial sum** of Eq.(1.3.1) is

$$S_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n \tag{1.3.2}$$

where $n = 0, 1, \cdots$.
Clearly, if we omit the term of S_n from Eq.(1.3.1), the remaining expression is

$$R_n(x) = a_{n+1}(x-x_0)^{n+1} + a_{n+2}(x-x_0)^{n+2} + \cdots \tag{1.3.3}$$

The expression is called the **remainder** of Eq.(1.3.1) after the term $a_n(x-x_0)^n$.

Convergence Interval, Radius of Convergence

The series Eq.(1.3.1) always converges at $x = x_0$, because then all its terms except for the first, a_0, are zero. If there are further values of x for which the series converges, these values form an interval, called the **convergence interval**. If this interval is finite, it has the midpoint x_0, so that it is of the form

$$|x-x_0| < R \tag{1.3.4}$$

and the series Eq.(1.3.1) converges for all x such that $|x-x_0| < R$ and diverges for all x such that $|x-x_0| > R$.
The number R is called the **radius of convergence** of Eq.(1.3.1).
The convergence interval may sometimes be infinite, that is, Eq.(1.3.1) converges for all x.

Real Analytic Functions

A real function $f(x)$ is called **analytic** at a point $x = x_0$ if it can be represented by a power series in powers of $x - x_0$ with radius of convergence $R > 0$.

Existence of Power Series Solutions

If p, q and r in an equation

$$y'' + p(x)y' + q(x)y = r(x) \tag{1.3.5}$$

are analytic at $x = x_0$, then every solution of Eq.(1.3.5) is analytic at $x = x_0$ and can thus be represented by a power series in powers of $x - x_0$ with radius of convergence $R > 0$.

Hence the same is true if $\tilde{h}$, $\tilde{p}$, $\tilde{q}$, and $\tilde{r}$ in an equation

$$\tilde{h}(x)y'' + \tilde{p}(x)y' + \tilde{q}(x)y = \tilde{r}(x) \tag{1.3.6}$$

are analytic at $x = x_0$ and $\tilde{h}(x_0) \neq 0$.

1.3.2 Frobenius Method

A point $x = x_0$ at which the coefficients p and q of an equation

$$y'' + p(x)y' + q(x)y = 0 \tag{1.3.7}$$

are analytic is called a **regular point** of the equation. A point that is not a regular point is called a **singular point** of the equation.

Frobenius Method

Any differential equation of the form

$$y'' + \frac{b(x)}{x}y' + \frac{c(x)}{x^2}y = 0 \tag{1.3.8}$$

where the functions $b(x)$ and $c(x)$ are analytic at $x = 0$, has at least one solution that can be represented in the form

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m = x^r (a_0 + a_1 x + a_2 x^2 + \cdots) \tag{1.3.9}$$

where the exponent r may be any (real or complex) number (and r is chosen so that $a_0 \neq 0$).

The equation also has a second solution (such that these two solutions are linearly independent) that may be similar to Eq.(1.3.9) (with a different r and different coefficients) or may contain a logarithmic term.

To solve Eq.(1.3.8), we write in the somewhat more convenient form

$$x^2 y'' + x b(x) y' + c(x) y = 0 \tag{1.3.10}$$

We first expand $b(x)$ and $c(x)$ in power series,

$$b(x) = b_0 + b_1 x + b_2 x^2 + \cdots, \quad c(x) = c_0 + c_1 x + c_2 x^2 + \cdots \tag{1.3.11}$$

Then we differentiate Eq.(1.3.9) term by term, finding

$$\begin{aligned} y'(x) &= \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1} = x^{r-1} \left[r a_0 + (r+1) a_1 x + \cdots \right] \\ y''(x) &= \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-2} = x^{r-2} \left[r(r-1) a_0 + (r+1) r a_1 x + \cdots \right] \end{aligned} \tag{1.3.12}$$

By inserting all these series into Eq.(1.3.10) we readily obtain

$$x^r \left[r(r-1) a_0 + \cdots \right] + (b_0 + b_1 x + \cdots) x^r (r a_0 + \cdots) + (c_0 + c_1 x + \cdots) x^r (a_0 + a_1 x + \cdots) = 0 \tag{1.3.13}$$

We now equate the sum of the coefficients of each power of x to zero. This yields a system of equations involving the unknown coefficients a_m. The smallest power is x^r, and the corresponding equation is

$$\left[r(r-1) + b_0 r + c_0 \right] a_0 = 0 \tag{1.3.14}$$

Since by assumption $a_0 \neq 0$, the expression in the brackets must be zero.
This give

$$r(r-1) + b_0 r + c_0 = 0 \tag{1.3.15}$$

This important quadratic equation is called the **indicial equation** of the differential equation Eq.(1.3.8).

Let r_1 and r_2 be the roots of the indicial equation Eq.(1.3.15). Then we have the following three cases.

(1) Distinct roots not differing by an integer

A basis is

$$y_1(x) = x^{r_1} (a_0 + a_1 x + a_2 x^2 + \cdots) \tag{1.3.16}$$

and

$$y_2(x) = x^{r_2} (A_0 + A_1 x + A_2 x^2 + \cdots) \tag{1.3.17}$$

with coefficients obtained successively from Eq.(1.3.13) with $r = r_1$ and $r = r_2$, respectively.

(2) Double root $r_1 = r_2 = r$

A basis is

$$y_1(x) = x^r (a_0 + a_1 x + a_2 x^2 + \cdots) \qquad [r = \tfrac{1}{2}(1 - b_0)] \tag{1.3.18}$$

(of the same general form as before) and

$$y_2(x) = y_1(x) \ln x + x^r (A_1 x + A_2 x^2 + \cdots) \qquad (x > 0) \tag{1.3.19}$$

(3) Roots differing by an integer

A basis is

$$y_1(x) = x^{r_1} (a_0 + a_1 x + a_2 x^2 + \cdots) \tag{1.3.20}$$

(of the same general form as before) and

$$y_2(x) = k y_1(x) \ln x + x^{r_2} (A_0 + A_1 x + A_2 x^2 + \cdots) \tag{1.3.21}$$

where the roots are so denoted that $r_1 - r_2 > 0$ and k may turn out to be zero.

Example

Solve the following differential equation.

$$xy'' + 2y' + 4xy = 0 \tag{1.3.22}$$

Solution

We apply the Frobenius method. A solution of this equation can be represented in the form

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} a_m x^{m+r} \qquad (a_0 \neq 0) \tag{1.3.23}$$

Substituting Eq.(1.3.23) and its derivatives into Eq.(1.3.22), we obtain

$$\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r-1} + 2\sum_{m=0}^{\infty}(m+r)a_m x^{m+r-1} + 4\sum_{m=0}^{\infty} a_m x^{m+r+1} = 0 \tag{1.3.24}$$

By equating the sum of the coefficient of the smallest power x^{r-1} to zero, we obtain the indicial equation

$$r(r-1)a_0 + 2ra_0 = 0\,, \qquad \text{thus} \qquad r^2 + r = 0 \tag{1.3.25}$$

The roots are

$$r_1 = 0, \quad r_2 = -1 \tag{1.3.26}$$

By equating the sum of the coefficient of x^r to zero, we obtain

$$(r+1)ra_1 + 2(r+1)a_1 = 0\,, \qquad \text{thus} \qquad (r^2+3r+2)a_1 = 0 \tag{1.3.27}$$

By equating the sum of the coefficient of x^{r+s+1} to zero, we obtain the recursion formula

$$(r+s+2)(r+s+1)a_{s+2} + 2(r+s+2)a_{s+2} + 4a_s = 0\,, \text{ thus } \; a_{s+2} = \frac{-4}{(s+r+2)(s+r+3)}a_s \quad (s = 0,1,\cdots) \tag{1.3.28}$$

We determine a first solution $y_1(x)$ corresponding to $r_1 = 0$. For $r = r_1$, Eq.(1.3.28) becomes

$$a_{s+2} = \frac{-4}{(s+2)(s+3)}a_s \qquad (s = 0,1,\cdots) \tag{1.3.29}$$

For $r = r_1 = 0$, Eq.(1.3.27) yields $a_1 = 0$. It follows that $a_3 = 0,\ a_5 = 0, \cdots$, successively.
We obtain for the other coefficients

$$a_2 = \frac{-4}{3!}a_0, \quad a_4 = \frac{-4}{5\cdot 4}a_2 = \frac{(-4)^2}{5!}a_0, \quad a_6 = \frac{-4}{7\cdot 6}a_4 = \frac{(-4)^3}{7!}a_0, \cdots \tag{1.3.30}$$

and in general,

$$a_{2m} = \frac{(-4)^m}{(2m+1)!}a_0 \qquad (m = 0,1,\cdots) \tag{1.3.31}$$

Hence, the first solution is

$$y_1 = x^0 \sum_{m=0}^{\infty} a_{2m}x^{2m} = a_0 \sum_{m=0}^{\infty}\frac{(-4)^m}{(2m+1)!}x^{2m} = a_0 \frac{1}{x}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}(2x)^{2m+1} = a_0 \frac{\sin 2x}{x} \tag{1.3.32}$$

Because y_1 is a familiar function, we get a second independent solution y_2 by the method of reduction of order. Substituting $y_2 = uy_1$ and its derivatives into the standard equation of Eq.(1.3.22),

$$y'' + 2x^{-1}y' + 4y = 0 \tag{1.3.33}$$

we get

$$u'' + \left(\frac{2y_1'}{y_1} + 2x^{-1}\right)u' = 0 \tag{1.3.34}$$

This gives

$$u' = \frac{1}{y_1^2}e^{-\int 2x^{-1}dx} = \frac{x^2}{\sin^2 2x}e^{-2\ln|x|} = \frac{x^2}{\sin^2 2x}x^{-2} = \frac{1}{\sin^2 2x} \tag{1.3.35}$$

Thus

$$u = \int \frac{1}{\sin^2 2x}dx = -\frac{1}{2}\cot 2x \tag{1.3.36}$$

Hence

$$y_2 = uy_1 = \frac{\sin 2x}{x}\cot 2x = \frac{\cos 2x}{x} \tag{1.3.37}$$

The answer is

$$y = C_1\frac{\sin 2x}{x} + C_2\frac{\cos 2x}{x} \tag{1.3.38}$$

Now, we find a second solution by using the power series. The roots of the indicial equation, $r_1 = 0,\ r_2 = -1$, differ by integer. Then, we start from a second solution expressed in the form

$$y_2(x) = ky_1(x)\ln x + x^{r_2}\sum_{m=0}^{\infty} A_m x^m = ky_1(x)\ln x + x^{-1}\sum_{m=0}^{\infty} A_m x^m \tag{1.3.39}$$

Substituting Eq.(1.3.39), Eq.(1.3.32) and their derivatives into Eq.(1.3.22), we obtain

$$k(2y_1' + x^{-1}y_1) + \sum_{m=0}^{\infty} m(m-1)A_m x^{m-2} + 4\sum_{m=0}^{\infty} A_m x^m = 0 \tag{1.3.40}$$

thus

$$k\sum_{m=0}^{\infty}(4m+1)a_{2m}x^{2m-1} + \sum_{m=0}^{\infty} m(m-1)A_m x^{m-2} + 4\sum_{m=0}^{\infty} A_m x^m = 0 \tag{1.3.41}$$

By equating the sum of the coefficient of the smallest power x^{-1} to zero, we have

$$ka_0 = 0 \tag{1.3.42}$$

Because of $a_0 \neq 0$, we obtain

$$k = 0 \tag{1.3.43}$$

By equating the sum of the coefficient of x^s to zero, we obtain the recursion formula

$$(s+2)(s+2-1)A_{s+2} + 4A_s = 0\,, \quad \text{thus} \quad A_{s+2} = \frac{-4}{(s+1)(s+2)}A_s \quad (s = 0,1,\cdots) \tag{1.3.44}$$

From this we get successively

$$\begin{aligned} &A_2 = \frac{-4}{2!}A_0, \quad A_4 = \frac{-4}{4\cdot 3}A_2 = \frac{(-4)^2}{4!}A_0, \quad A_6 = \frac{-4}{6\cdot 5}A_4 = \frac{(-4)^3}{6!}A_0, \cdots \\ &A_3 = \frac{-4}{3!}A_1, \quad A_5 = \frac{-4}{5\cdot 4}A_3 = \frac{(-4)^2}{5!}A_1, \quad A_5 = \frac{-4}{7\cdot 6}A_3 = \frac{(-4)^3}{7!}A_1, \cdots \end{aligned} \tag{1.3.45}$$

and in general,

$$A_{2m} = \frac{(-4)^m}{(2m)!}A_0, \quad A_{2m+1} = \frac{(-4)^m}{(2m+1)!}A_1 \qquad (m = 0,1,\cdots) \tag{1.3.46}$$

Hence,

$$y_2(x) = x^{-1}\left(\sum_{m=0}^{\infty} A_{2m}x^{2m} + \sum_{m=0}^{\infty} A_{2m+1}x^{2m+1}\right) = A_0\sum_{m=0}^{\infty}\frac{(-4)^m}{(2m)!}x^{2m-1} + A_1\sum_{m=0}^{\infty}\frac{(-4)^m}{(2m+1)!}x^{2m} \tag{1.3.47}$$

Here, from Eq.(1.3.32), the even power series gives

$$A_1\sum_{m=0}^{\infty}\frac{(-4)^m}{(2m+1)!}x^{2m} = \frac{A_1}{a_0}y_1 \tag{1.3.48}$$

Then, taking $A_1 = 0$, a second independent solution is

$$y_2(x) = A_0\sum_{m=0}^{\infty}\frac{(-4)^m}{(2m)!}x^{2m-1} = A_0\frac{1}{x}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)!}(2x)^{2m} = A_0\frac{\cos 2x}{x} \tag{1.3.49}$$

1.3.3 Legendre's Equation. Legendre Polynomials $P_n(x)$

Legendre's differential equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \tag{1.3.50}$$

where the parameter n in Eq.(1.3.50) is a given real number. The solutions of Eq.(1.3.50) are called **Legendre functions**.

The coefficients of Eq.(1.3.50) are analytic at $x = 0$ and $\tilde{h}(x) = 1 - x^2 \neq 0$ at $x = 0$. Hence, every solution of Eq.(1.3.50) is some power series

$$y(x) = \sum_{m=0}^{\infty} a_m x^m \tag{1.3.51}$$

Substituting Eq.(1.3.51) and its derivatives into Eq.(1.3.50), we obtain a general solution of Eq.(1.3.50).

In many applications the parameter n in Legendre's differential equation will be a nonnegative integer. Then the following polynomial which is a solution of Legendre's differential equation Eq.(1.3.50) is called **Legendre polynomial** of degree n and is denoted by $P_n(x)$.

$$P_n(x) = \sum_{m=0}^{M}(-1)^m\frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!}x^{n-2m} = \frac{(2n)!}{2^n(n!)^2}x^n - \frac{(2n-2)!}{2^n 1!(n-1)!(n-2)!}x^{n-2} + -\cdots \tag{1.3.52}$$

where $M = n/2$ or $(n-1)/2$, whichever is an integer.

In particular,

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= (3x^2-1)/2 \\ P_3(x) &= (5x^3-3x)/2 \\ P_4(x) &= (35x^4-30x^2+3)/8 \\ P_5(x) &= (63x^5-70x^3+15x)/8 \end{aligned} \qquad (1.3.53)$$

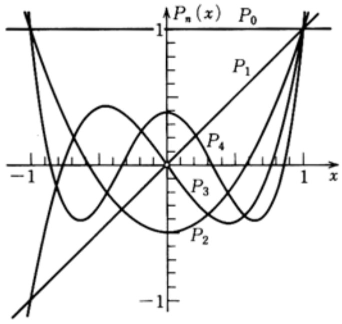

Fig.1.3.1 Legendre polynomials

1.3.4 Bessel's Equation. Bessel Functions $J_\nu(x)$ and $Y_\nu(x)$

Bessel's differential equation

$$x^2y'' + xy' + (x^2-\nu^2)y = 0 \qquad (1.3.54)$$

where the parameter ν is a given nonnegative real number.

We can solve it by the Frobenius Method. Substituting a solution of the form

$$y(x) = \sum_{m=0}^{\infty} a_m x^{m+r} \qquad (a_0 \neq 0) \qquad (1.3.55)$$

and its derivatives into Bessel's differential equation Eq.(1.3.54), we obtain a solution of Eq.(1.3.54).

If ν is not an integer, a general solution of Bessel's differential equation for all $x \neq 0$ is

$$y(x) = c_1 J_\nu(x) + c_2 J_{-\nu}(x) \qquad (1.3.56)$$

where $J_\nu(x)$ is the function called the **Bessel function of the first kind of order** ν of the form

$$J_\nu(x) = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu} m!\Gamma(\nu+m+1)} \qquad (1.3.57)$$

and $\Gamma(\nu)$ is the **gamma function** defined by the integral

$$\Gamma(\nu) = \int_0^\infty e^{-t} t^{\nu-1} dt \qquad (\nu > 0) \qquad (1.3.58)$$

In general $\Gamma(n+1) = n! \qquad (n = 0, 1, \cdots)$ (1.3.59)

hence $$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!(n+m)!} \qquad (1.3.60)$$

But if ν is an integer n, then Eq.(1.3.56) is not a general solution because $J_n(x)$ and $J_{-n}(x)$ are linearly dependent, that is,

$$J_{-n}(x) = (-1)^n J_n(x) \qquad (n = 1, 2, \cdots) \qquad (1.3.61)$$

A general solution of Bessel's differential equation for all values of ν is

$$y(x) = c_1 J_\nu(x) + c_2 Y_\nu(x) \qquad (1.3.62)$$

where $Y_\nu(x)$ is the function called the **Bessel function of the second kind of order** ν or **Neumann's function of order** ν and is defined for all ν by the formula

$$\begin{aligned} Y_\nu(x) &= \frac{1}{\sin\nu\pi}\left[J_\nu(x)\cos\nu\pi - J_{-\nu}(x)\right] \\ Y_n(x) &= \lim_{\nu\to n} Y_\nu(x) \end{aligned} \qquad (1.3.63)$$

$J_\nu(x)$ and $Y_\nu(x)$ are linearly independent. Also $J_n(x)$ and $Y_n(x)$ are linearly independent.

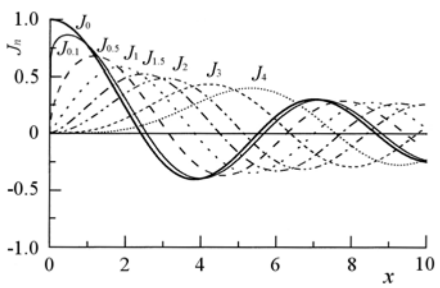

Fig.1.3.2 Bessel functions of the first kind

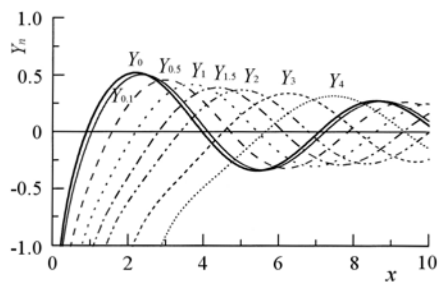

Fig.1.3.3 Bessel functions of the second kind

Problem

Apply the power series method or the Frobenius method to the following differential equations. Try to identify the series obtained as expansions of known functions.

(1) $y' = 2xy$

(2) $y'' + 4y = 0$

(3) $y'' - y = 0$

(4) $4x^2 y'' - 3y = 0$

(5) $x^2 y'' - 3xy' + 4y = 0$

(6) $xy'' + 3y' + 4x^3 y = 0$

(7) $x(1-x)y'' + 2(1-2x)y' - 2y = 0$

(8) $2x(x-1)y'' - (x+1)y' + y = 0$

1.4 Orthogonal Expansions

1.4.1 Orthogonality

Functions $y_1, y_2, \cdots$ defined on some interval $a \le x \le b$ are called **orthogonal** on $a \le x \le b$ with respect to a **weight function** $p(x) > 0$ if

$$\int_a^b p(x) y_m(x) y_n(x)\, dx = 0 \qquad \text{for } m \ne n \tag{1.4.1}$$

The **norm** $\|y_m\|$ of y_m is defined by

$$\|y_m\| = \sqrt{\int_a^b p(x) y_m^2(x)\, dx} \tag{1.4.2}$$

The functions are called **orthonormal** on $a \le x \le b$ if they are orthogonal on $a \le x \le b$ and all have norm 1.
Some orthogonal functions are listed in Table 1.4.1.

Table 1.4.1 Examples of orthogonal functions

Function	Notation	Differential Equation	Interval	Weight
Trigonometric function	$1, \cos nx, \sin nx$	$y'' + \lambda y = 0 \quad (\lambda = n^2)$	$[-\pi, \pi]$	1
Legendre	$P_n(x)$	$(1-x^2)y'' - 2xy' + n(n+1)y = 0$	$[-1, 1]$	1
Hermite	$H_n(x)$	$y'' - 2xy' + 2ny = 0$	$(-\infty, \infty)$	e^{-x^2}
Chebyshev (first kind)	$T_n(x)$	$(1-x^2)y'' - xy' + n^2 y = 0$	$[-1, 1]$	$1/\sqrt{1-x^2}$
Laguerre	$L_n(x)$	$xy'' + (1-x)y' + ny = 0$	$[0, \infty)$	e^{-x}
Bessel	$J_n(x)$	$x^2 y'' + xy' + (x^2 - n^2)y = 0$	$[0, R]$	x/λ_{mn}

[R is a given number, $\lambda_{mn} = \alpha_{mn}/R$, α_{mn} is the mth real zero of $J_n(x)$]

1.4.2 Orthogonal Expansions

The **orthogonal expansion** or **generalized Fourier series** of a given function $f(x)$ in term of an orthogonal set $y_0, y_1, \cdots$ with respect to weight $p(x)$ on an interval $a \le x \le b$ is

$$f(x) = \sum_{n=0}^{\infty} a_n y_n(x) \tag{1.4.3}$$

where
$$a_n = \frac{1}{\|y_n\|^2} \int_a^b p(x) f(x) y_n(x)\, dx, \quad (n = 0, 1, \cdots) \tag{1.4.4}$$

$a_0, a_1, \cdots$ are called the **Fourier constants** of $f(x)$ with respect to $y_0, y_1, \cdots$.

If we multiply both side of Eq.(1.4.3) by $p(x) y_m(x)$ (m fixed) and then integrate over $a \le x \le b$, we obtain,

$$\int_a^b p(x) f(x) y_m(x)\, dx = \int_a^b p(x) \left(\sum_{n=0}^{\infty} a_n y_n(x) \right) y_m(x)\, dx = \sum_{n=0}^{\infty} a_n \int_a^b p(x) y_n(x) y_m(x)\, dx \tag{1.4.5}$$

Because of the orthogonality, all the integrals on the right are zero, except when $n = m$;
then $\int_a^b p(x) y_m^2(x)\, dx = \|y_m\|^2$, so that the above equation reduces to

$$\int_a^b p(x) f(x) y_m(x)\, dx = a_m \|y_m\|^2 \tag{1.4.6}$$

Writing n for m, to be in agreement with the notation in Eq.(1.4.3), we get Eq.(1.4.4).

Example

Obtain the first few terms of the expansion of $f(x)$ in term of Legendre polynomials and Chebyshev polynomials and graph their partial sums.

$$f(x) = \begin{cases} 0 & \text{if } -1 < x < 0 \\ x & \text{if } \ 0 < x < 1 \end{cases} \tag{1.4.7}$$

Solution

We get the coefficients of the Fourier-Legendre series

$$a_0 = \frac{1}{\|P_0\|^2}\int_{-1}^{1} f(x)P_0 dx = \frac{1}{2}\int_0^1 x dx = \frac{1}{4}$$
$$a_1 = \frac{1}{\|P_1\|^2}\int_{-1}^{1} f(x)P_1 dx = \frac{3}{2}\int_0^1 x\cdot x dx = \frac{1}{2}$$
$$a_2 = \frac{1}{\|P_2\|^2}\int_{-1}^{1} f(x)P_2 dx = \frac{5}{2}\int_0^1 x\frac{1}{2}(3x^2-1)dx = \frac{5}{16} \qquad (1.4.8)$$
$$a_3 = \frac{1}{\|P_3\|^2}\int_{-1}^{1} f(x)P_3 dx = \frac{7}{2}\int_0^1 x\frac{1}{2}(5x^3-3x)dx = 0$$

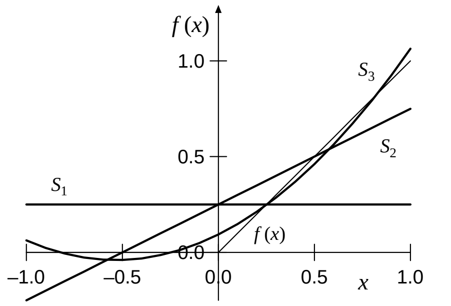

Fig.1.4.1 Partial sums of the Fourier-Legendre series of $f(x)$

Hence, the Fourier-Legendre series of $f(x)$ is

$$f(x) = \frac{1}{4}P_0 + \frac{1}{2}P_1 + \frac{5}{16}P_2 + \cdots \qquad (1.4.9)$$

Next, we obtain the coefficients of the Fourier-Chebyshev series

$$a_0 = \frac{1}{\|T_0\|^2}\int_{-1}^{1} \frac{T_0}{\sqrt{1-x^2}} f(x)dx = \frac{1}{\pi}\int_0^1 \frac{1}{\sqrt{1-x^2}} x dx = \frac{1}{\pi}$$
$$a_1 = \frac{1}{\|T_1\|^2}\int_{-1}^{1} \frac{T_1}{\sqrt{1-x^2}} f(x)dx = \frac{2}{\pi}\int_0^1 \frac{x}{\sqrt{1-x^2}} x dx = \frac{1}{2}$$
$$a_2 = \frac{1}{\|T_2\|^2}\int_{-1}^{1} \frac{T_2}{\sqrt{1-x^2}} f(x)dx = \frac{2}{\pi}\int_0^1 \frac{2x^2-1}{\sqrt{1-x^2}} x dx = \frac{2}{3\pi} \qquad (1.4.10)$$
$$a_3 = \frac{1}{\|T_3\|^2}\int_{-1}^{1} \frac{T_3}{\sqrt{1-x^2}} f(x)dx = \frac{2}{\pi}\int_0^1 \frac{4x^3-3x}{\sqrt{1-x^2}} x dx = 0$$

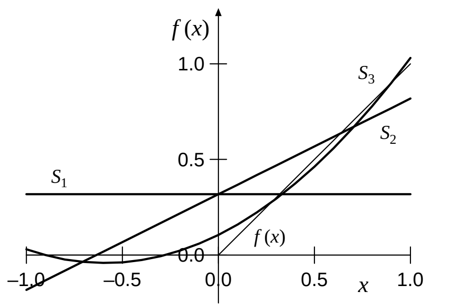

Fig.1.4.2 Partial sums of the Fourier-Chebyshev series of $f(x)$

Hence, the Fourier-Chebyshev series of $f(x)$ is

$$f(x) = \frac{1}{\pi}T_0 + \frac{1}{2}T_1 + \frac{2}{3\pi}T_2 + \cdots \qquad (1.4.11)$$

Figure 1.4.1 and 1.4.2 show the first few partial sums of the Fourier-Legendre series and the Fourier-Chebyshev series of $f(x)$. In these figures, S_n, $(n = 1, 2, \cdots)$ denotes the partial sum of the first n terms.

The Fourier series using trigonometric functions as orthogonal functions will be discussed in the next section.

Problem

Obtain the first few terms of the expansion of $f(x)$ in term of Legendre polynomials and graph their partial sums.

(1) $f(x) = x$

(2) $f(x) = x^2$

(3) $f(x) = x^3$

(4) $f(x) = 5x^3 + 3x^2 - 3x - 1$

(5) $f(x) = \dfrac{1}{x+2}$ if $-1 < x < 1$

(6) $f(x) = \sqrt{1-x^2}$ if $-1 < x < 1$

(7) $f(x) = |x|$ if $-1 < x < 1$

(8) $f(x) = e^x$ if $-1 < x < 1$

(9) $f(x) = \sin \pi x$ if $-1 < x < 1$

(10) $f(x) = 4\cos \pi x + \cos 4\pi x$ if $-1 < x < 1$

(11) $f(x) = \begin{cases} 0 & \text{if } -1 < x < 0 \\ 1 & \text{if } 0 < x < 1 \end{cases}$

(12) $f(x) = \begin{cases} 1 & \text{if } -1 < x < 0 \\ x & \text{if } 0 < x < 1 \end{cases}$

(13) $f(x) = \begin{cases} 0 & \text{if } -1 < x < -1/2 \\ 1 & \text{if } -1/2 < x < 1 \end{cases}$

(14) $f(x) = \begin{cases} 1+x & \text{if } -1 < x < 0 \\ 1-x & \text{if } 0 < x < 1 \end{cases}$

(15) $f(x) = \begin{cases} 0 & \text{if } -1 < x < 0 \\ \sin \pi x & \text{if } 0 < x < 1 \end{cases}$

(16) $f(x) = |\sin \pi x|$ if $-1 < x < 1$