2. Fourier Series, Fourier Integrals and Fourier Transforms

The Fourier series are used for the analysis of the periodic phenomena, which often appear in physics and engineering. The Fourier integrals and Fourier transforms extend the ideas and techniques of the Fourier series to the non-periodic phenomenon. The Fourier transform is commonly used to transform a problem from the "time domain" into the "frequency domain" in which the amplitude and the phase are given as a function of frequency.

2.1 Fourier Series

2.1.1 Periodic Functions

If a function $f(x)$ is defined for all real x and if there is some positive number p such that

$$f(x+p) = f(x) \qquad \text{for all } x \tag{2.1.1}$$

it is called **periodic**. The number p is called a **period** of $f(x)$.

From Eq.(2.1.1), for any integer n,

$$f(x+np) = f(x) \qquad \text{for all } x \tag{2.1.2}$$

Hence $2p,\ 3p,\ 4p,\cdots$ are also period of $f(x)$.

If a periodic function $f(x)$ has a smallest period $p(>0)$, this is often called the **primitive period** of $f(x)$.

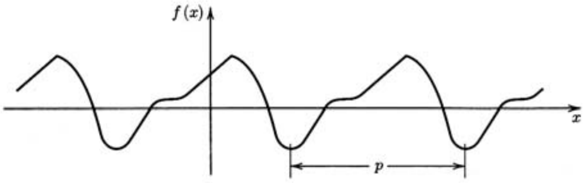


Fig.2.1.1 Periodic function

If $f(x)$ and $g(x)$ have period p, then the function

$$h(x) = af(x) + bg(x) \qquad (a, b \quad \text{constant}) \tag{2.1.3}$$

also has the period p.

2.1.2 Fourier Series of a Periodic Function with Period 2π

Let us assume that $f(x)$ is a periodic function of period 2π that can be represented by a trigonometric series.

Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right) \tag{2.1.4}$$

Fourier coefficients (given by the **Euler formulas**)

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad (n = 1, 2, \cdots) \tag{2.1.5}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx, \quad (n = 1, 2, \cdots)$$

It can be obtained from the orthogonality of trigonometric system on an interval of length 2π in Sec. 1.4.

$$\int_{-\pi}^{\pi}\cos nx\cos mx\,dx = \begin{cases} 0 & (n \neq m) \\ \pi & (n = m) \end{cases}, \quad \int_{-\pi}^{\pi}\sin nx\sin mx\,dx = \begin{cases} 0 & (n \neq m) \\ \pi & (n = m) \end{cases} \tag{2.1.6}$$

$$\int_{-\pi}^{\pi}\cos nx\sin mx\,dx = 0, \quad \int_{-\pi}^{\pi}\cos nx\,dx = 0, \quad \int_{-\pi}^{\pi}\sin nx\,dx = 0 \quad \text{for any integers } n \text{ and } m.$$

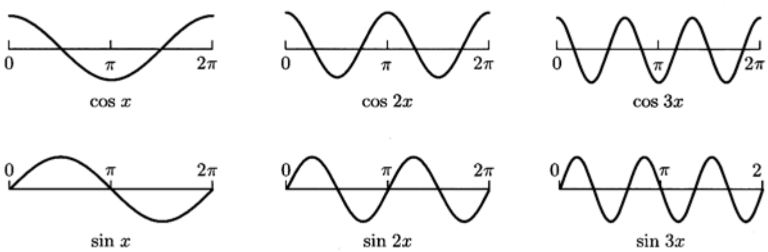


Fig.2.1.2 Cosine and sine functions having the period 2π

If a periodic function $f(x)$ with period 2π is piecewise continuous in the interval $-\pi \le x \le \pi$ and has a left-hand derivative and right-hand derivative at each point of that interval, then the Fourier series Eq.(2.1.4) of $f(x)$ is convergent. Its sum is $f(x)$, except at a point x_0 at which $f(x)$ is discontinuous and the sum of the series is the average of the left- and right-hand limits of $f(x)$ at x_0.

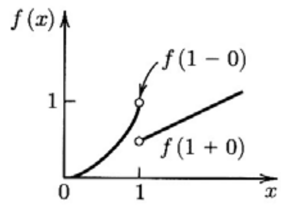


Fig.2.1.3 Left- and right-hand limits

Parseval's Theorem

If a function $f(x)$ is square-integrable on an interval $-\pi \le x \le \pi$, then

$$a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)^2\,dx \tag{2.1.7}$$

2.1.3 Fourier Series of a Function of Any Period $p = 2L$

If a function $f(x)$ has period $p = 2L$, then

Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right) \tag{2.1.8}$$

Fourier coefficients

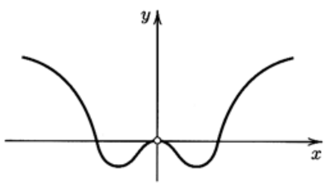
$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx, \quad (n = 1, 2, \cdots) \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx, \quad (n = 1, 2, \cdots) \end{aligned} \tag{2.1.9}$$

($\because$ Let $x = \frac{\pi}{L}v$ in Eq.(2.1.4) and Eq.(2.1.5))

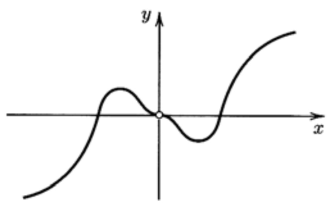
2.1.4 Even and Odd Functions

A function $g(x)$ is **even** if $g(-x) = g(x)$ for all x.

A function $h(x)$ is **odd** if $h(-x) = -h(x)$ for all x.



Even function



Odd function

Fig.2.1.4 Even function and odd function

If $g(x)$ is an even function, then $\int_{-L}^{L} g(x)dx = 2\int_0^L g(x)dx$.

If $h(x)$ is an odd function, then $\int_{-L}^{L} h(x)dx = 0$.

The product of an even function and an odd function is odd.

The product of an even function and an even function, or that of an odd function and an odd function is even.

Fourier Series of an Even Function of Period $p = 2L$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x \qquad (f(x) \text{ is even function}) \qquad \textbf{Fourier cosine series} \tag{2.1.10}$$

with coefficients

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx, \quad (n = 1, 2, \cdots) \quad (b_n = 0) \tag{2.1.11}$$

Fourier Series of an Odd Function of Period $p = 2L$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x \qquad (f(x) \text{ is odd function}) \qquad \textbf{Fourier sine series} \tag{2.1.12}$$

with coefficients

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx, \quad (n = 1, 2, \cdots) \quad (a_0 = 0,\ a_n = 0) \tag{2.1.13}$$

2.1.5 Complex Fourier Series

The Fourier series of a periodic function $f(x)$ of period 2π

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right) \tag{2.1.14}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx, \quad (n = 1, 2, \cdots) \tag{2.1.15}$$

can be written in complex form.
Using the Euler formula, we get

$$e^{inx} = \cos nx + i\sin nx, \quad e^{-inx} = \cos nx - i\sin nx \tag{2.1.16}$$

$$\cos nx = \frac{1}{2}\left(e^{inx} + e^{-inx}\right), \quad \sin nx = \frac{1}{2i}\left(e^{inx} - e^{-inx}\right) = \frac{i}{2}\left(-e^{inx} + e^{-inx}\right) \tag{2.1.17}$$

With this, Eq.(2.1.14) becomes

$$\begin{aligned} f(x) &= a_0 + \sum_{n=1}^{\infty}\left\{\frac{1}{2}a_n\left(e^{inx} + e^{-inx}\right) + \frac{1}{2i}b_n\left(e^{inx} - e^{-inx}\right)\right\} \\ &= a_0 + \sum_{n=1}^{\infty}\left\{\frac{1}{2}\left(a_n - ib_n\right)e^{inx} + \frac{1}{2}\left(a_n + ib_n\right)e^{-inx}\right\} \end{aligned} \tag{2.1.18}$$

If we introduce the notations,

$$\begin{aligned} c_0 &= a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ c_n &= \frac{1}{2}\left(a_n - ib_n\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\left(\cos nx - i\sin nx\right)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \\ c_{-n} &= \frac{1}{2}\left(a_n + ib_n\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\left(\cos nx + i\sin nx\right)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}dx \end{aligned} \tag{2.1.19}$$

we obtain

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \; c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx, \quad (n = 0, \pm 1, \pm 2, \cdots) \tag{2.1.20}$$

2.1.6 Complex Fourier Series of Function $f(x)$ of Period $p = 2L$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}, \; c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}dx, \quad (n = 0, \pm 1, \pm 2, \cdots) \tag{2.1.21}$$

Parseval's Theorem

If a function $f(x)$ is square-integrable on an interval $-L \le x \le L$, then

$$\sum_{n=-\infty}^{\infty} |\,c_n|^2 = \frac{1}{2L}\int_{-L}^{L} f(x)^2\,dx \tag{2.1.22}$$

Summary

Fourier series of a periodic function $f(x)$ of period $p = 2L$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} \tag{2.1.23}$$

Fourier coefficients

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx, \quad (n = 1, 2, \cdots) \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx, \quad (n = 1, 2, \cdots) \\ c_n &= \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}dx, \quad (n = 0, \pm 1, \pm 2, \cdots) \end{aligned} \tag{2.1.24}$$

Example

Find the Fourier series of the following periodic function of period $p = 2$, and graph their partial sums.

$$f(x) = \begin{cases} 0 & \text{if } -1 < x < 0 \\ x & \text{if } \ 0 < x < 1 \end{cases} \tag{2.1.25}$$

Solution

Since the period $p = 2L = 2$, we obtain by the Euler formulas

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx = \frac{1}{2}\int_0^1 xdx = \frac{1}{4}$$

$$a_n = \frac{1}{L}\int_{-1}^{1} f(x)\cos\frac{n\pi x}{L}dx = \int_0^1 x\cos n\pi x dx$$

$$= \frac{\sin n\pi}{n\pi} + \frac{\cos n\pi - 1}{n^2\pi^2} = \begin{cases} \dfrac{-2}{n^2\pi^2} & \text{for odd } n \\ 0 & \text{for even } n \end{cases}$$

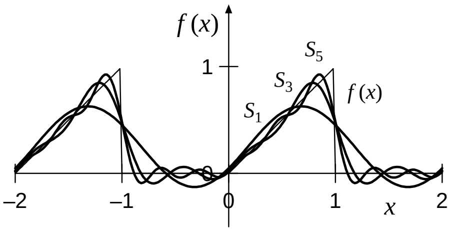


Fig.2.1.5 Partial sums of the Fourier series of $f(x)$

$$b_n = \frac{1}{L}\int_{-1}^{1} f(x)\sin\frac{n\pi x}{L}dx = \int_0^1 x\sin n\pi x dx = \frac{-\cos n\pi}{n\pi} + \frac{\sin n\pi}{n^2\pi^2} = \begin{cases} \dfrac{1}{n\pi} & \text{for odd } n \\ \dfrac{-1}{n\pi} & \text{for even } n \end{cases} \tag{2.1.26}$$

Hence, the Fourier series of $f(x)$ is

$$f(x) = \frac{1}{4} - \frac{2}{\pi^2}\left(\cos\pi x + \frac{1}{9}\cos 3\pi x + \frac{1}{25}\cos 5\pi x + \cdots\right) + \frac{1}{\pi}\left(\sin\pi x - \frac{1}{2}\sin 2\pi x + \frac{1}{3}\sin 3\pi x - \cdots\right) \tag{2.1.27}$$

Figure 2.1.5 shows the partial sum

$$S_n = \frac{1}{4} + \frac{1}{\pi^2}\sum_{m=1}^{n}\frac{(-1)^m - 1}{m^2}\cos m\pi x + \frac{1}{\pi}\sum_{m=1}^{n}\frac{-(-1)^m}{m}\sin m\pi x, \quad (n = 1, 2, \cdots) \tag{2.1.28}$$

This figure shows oscillations near the points of discontinuity of $f(x)$. We might expect that these oscillations disappear as n approaches infinity, but this is not true; with increasing n, they are shifted closer to the points of discontinuity. This unexpected behavior is known as the **Gibbs phenomenon**.

Next, we obtain the complex Fourier coefficients

$$\begin{aligned} c_0 &= a_0 = \frac{1}{4} \\ c_n &= \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}dx = \frac{1}{2}\int_0^1 x e^{-in\pi x}dx = \frac{1}{2n^2\pi^2}\left\{-1 + e^{-in\pi}(1 + in\pi)\right\} \quad (n = \pm 1, \pm 2, \cdots) \end{aligned} \tag{2.1.29}$$

Thus, we should often separately calculate only c_0 .

Problem

Find the Fourier series of the following functions, which are assumed to be periodic of the period $p = 2$, and graph their partial sums.

(1) $f(x) = x \quad (-1 < x < 1)$

(2) $f(x) = x^2 \quad (-1 < x < 1)$

(3) $f(x) = |x| \quad (-1 < x < 1)$

(4) $f(x) = e^x \quad (-1 < x < 1)$

(5) $f(x) = \sin\pi x$

(6) $f(x) = 4\cos\pi x + \cos 4\pi x$

(7) $f(x) = \begin{cases} 0 & \text{if } -1 < x < 0 \\ 1 & \text{if } \ 0 < x < 1 \end{cases}$

(8) $f(x) = \begin{cases} 1 & \text{if } -1 < x < 0 \\ x & \text{if } \ 0 < x < 1 \end{cases}$

(9) $f(x) = \begin{cases} 0 & \text{if } -1 < x < -1/2 \\ 1 & \text{if } -1/2 < x < 1 \end{cases}$

(10) $f(x) = \begin{cases} 1 + x & \text{if } -1 < x < 0 \\ 1 - x & \text{if } \ 0 < x < 1 \end{cases}$

(11) $f(x) = \begin{cases} 0 & \text{if } -1 < x < 0 \\ \sin\pi x & \text{if } \ 0 < x < 1 \end{cases}$

(12) $f(x) = |\sin\pi x| \quad (-1 < x < 1)$

2.2 Fourier Integrals and Fourier Transforms

In the previous section, we found that a periodic function can be represented by a Fourier series. We want to extend the method of Fourier series to nonperiodic functions. We see what happens to the periodic function of period $p = 2L$ if we let $L \to \infty$.

2.2.1 Fourier Integral

We consider any periodic function $f_L(x)$ of period $p = 2L$ that can be represented by a Fourier series

$$\begin{aligned} f_L(x) &= a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\omega_n x + b_n \sin\omega_n x\right) \\ &= \frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv + \frac{1}{L}\sum_{n=1}^{\infty}\left[\cos\omega_n x\int_{-L}^{L} f_L(v)\cos\omega_n v\,dv + \sin\omega_n x\int_{-L}^{L} f_L(v)\sin\omega_n v\,dv\right] \end{aligned} \tag{2.2.1}$$

where $\omega_n = n\pi/L$

We now set

$$\Delta\omega = \omega_{n+1} - \omega_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L} \tag{2.2.2}$$

Then $1/L = \Delta\omega/\pi$, and we may write Eq.(2.2.1) in the form

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[(\cos\omega_n x)\Delta\omega\int_{-L}^{L} f_L(v)\cos\omega_n v\,dv + (\sin\omega_n x)\Delta\omega\int_{-L}^{L} f_L(v)\sin\omega_n v\,dv\right] \tag{2.2.3}$$

This representation is valid for any fixed L , arbitrarily large, but finite.

We now let $L \to \infty$ and assume that the resulting nonperiodic function $f(x) = \lim_{L\to\infty} f_L(x)$ is absolutely integrable on the x-axis; that is, $\int_{-\infty}^{\infty} |f(x)|\,dx$ exists and it is finite.

Then $L \to \infty$, and the value of the first term on the right side of Eq.(2.2.3) approaches zero.

Also $\Delta\omega = \pi/L \to 0$ and it seems plausible that infinite series in Eq.(2.2.3) becomes an integral from $\omega = 0$ to $\omega = \infty$, which represents $f(x)$,

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\left[\cos\omega x\int_{-\infty}^{\infty} f(v)\cos\omega v\,dv + \sin\omega x\int_{-\infty}^{\infty} f(v)\sin\omega v\,dv\right]d\omega \tag{2.2.4}$$

If we introduce the notations

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v\,dv, \quad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v\,dv \tag{2.2.5}$$

we can write this in the form

$$f(x) = \int_{0}^{\infty}\left[A(\omega)\cos\omega x + B(\omega)\sin\omega x\right]d\omega \tag{2.2.6}$$

This is a representation of $f(x)$ by a **Fourier integral**.

This naive approach merely suggests the representation Eq.(2.2.6) but by no means establishes it.

Sufficient Conditions for the Fourier Integral

If $f(x)$ is piecewise continuous in every finite interval and has a right-hand derivative and a left-hand derivative at every point and if it is absolutely integrable on the x-axis, then $f(x)$ can be represented by a Fourier integral Eq.(2.2.6).

At a point where $f(x)$ is discontinuous the value of the Fourier integral equals the average of the left- and right-hand limits of $f(x)$ at that point.

Complex Fourier Integral

Similarly, we can derive the following complex Fourier integral from the complex Fourier series.

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)e^{i\omega(x-v)}\,dv\,d\omega \tag{2.2.7}$$

2.2.2 Fourier Transform

Writing the exponential function in Eq.(2.2.7) as a product of exponential functions, we have

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)e^{-i\omega v}dv\right]e^{i\omega x}d\omega \tag{2.2.8}$$

The expression in brackets is a function of ω, is denoted by $\hat{f}(\omega)$, and is called the **Fourier transform** of $f(x)$; writing $v=x$, we have

Fourier transform

$$\hat{f}(\omega)=\mathscr{F}\left[f(x)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx \tag{2.2.9}$$

Inverse Fourier transform

$$f(x)=\mathscr{F}^{-1}\left[\hat{f}(\omega)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega x}d\omega \tag{2.2.10}$$

Sufficient Conditions for Existence of the Fourier Transform

If $f(x)$ is piecewise continuous on every finite interval and it is absolutely integrable on the x-axis, then the Fourier transform of a function $f(x)$ exists.

Summary

Fourier integral of a function $f(x)$

$$f(x)=\int_{0}^{\infty}\left[A(\omega)\cos\omega x+B(\omega)\sin\omega x\right]d\omega=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}C(\omega)e^{i\omega x}d\omega \tag{2.2.11}$$

where

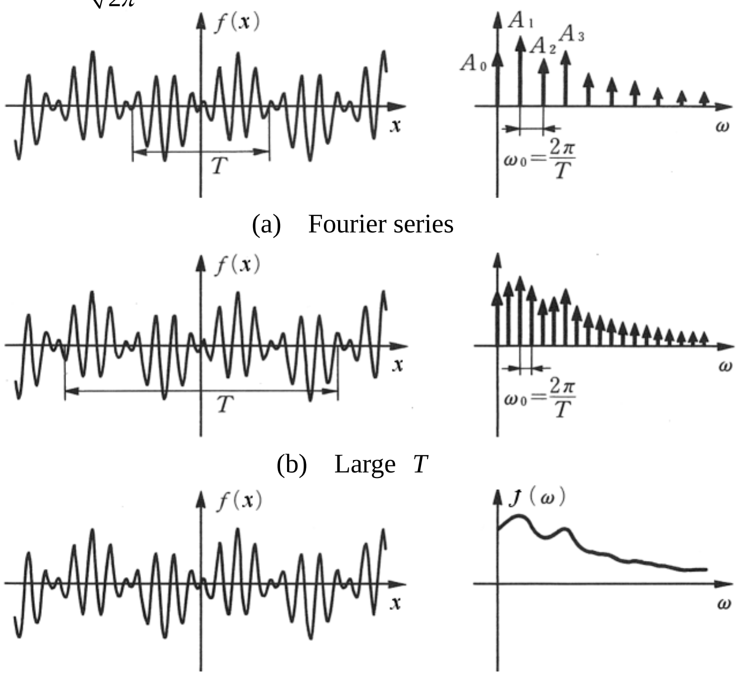
$$\begin{gathered}A(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x\,dx,\ B(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x\,dx\\ C(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx\end{gathered} \tag{2.2.12}$$

Fourier transform

$$\hat{f}(\omega)=\mathscr{F}\left[f(x)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx \tag{2.2.13}$$

Inverse Fourier transform

$$f(x)=\mathscr{F}^{-1}\left[\hat{f}(\omega)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega x}d\omega \tag{2.2.14}$$



(c) $T\to\infty$ (Fourier transform)

Fig.2.2.1 Fourier series and Fourier transform (period $p=T$)

Table 2.2.1 Formulas of the Fourier transform
(a and b are constants)

(1)	Linearity	$af(x)+bg(x)$	$a\hat{f}(\omega)+b\hat{g}(\omega)$
(2)	Derivative of function	$f^{(n)}(x)$	$(i\omega)^n\ \hat{f}(\omega)$
(3)	Shifting on the ω -axis	$e^{iax}f(x)$	$\hat{f}(\omega-a)$
(4)	Shifting on the x -axis	$f(x-a)$	$e^{-i\omega a}\hat{f}(\omega)$
(5)	Differentiation of transform	$x\,f(x)$	$i\,\hat{f}'(\omega)$
(6)	Scaling $(a\neq 0)$	$f(ax)$	$\frac{1}{\lvert a\rvert}\hat{f}(\frac{\omega}{a})$
(7)	Convolution	$(f*g)(x)=\int_{-\infty}^{\infty}f(p)g(x-p)\,dp$ $=\int_{-\infty}^{\infty}f(x-p)g(p)\,dp$	$\sqrt{2\pi}\,\hat{f}(\omega)\hat{g}(\omega)$
(8)	Duality	$\hat{f}(x)$	$f(-\omega)$

2.2.3 Formulas of the Fourier Transform

(1) Linearity (a and b are constants)

$$\begin{aligned}\mathcal{F}\left[af(x)+bg(x)\right]&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\{af(x)+bg(x)\}\,e^{-i\omega x}dx\\&=a\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-i\omega x}dx+b\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(x)e^{-i\omega x}dx\\&=a\mathcal{F}\left[f(x)\right]+b\mathcal{F}\left[g(x)\right]\end{aligned}\tag{2.2.15}$$

(2) Fourier transform of the derivative of a function

Let $f(x)$ be continuous on the x-axis and $f(x)\to 0$ as $\lvert x\rvert\to\infty$.

Furthermore, let $f'(x)$ be absolutely integrable on the x-axis. Then

$$\begin{aligned}\mathcal{F}\left[f'(x)\right]&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f'(x)e^{-i\omega x}dx=\frac{1}{\sqrt{2\pi}}\left\{\left[f(x)e^{-i\omega x}\right]_{-\infty}^{\infty}-(-i\omega)\int_{-\infty}^{\infty}f(x)e^{-i\omega x}dx\right\}\\&=0+i\omega\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-i\omega x}dx=i\omega\mathcal{F}\left[f(x)\right]\end{aligned}\tag{2.2.16}$$

Two successive applications of the above equation give

$$\mathcal{F}\left[f''(x)\right]=i\omega\mathcal{F}\left[f'(x)\right]=(i\omega)^2\mathcal{F}\left[f(x)\right]=-\omega^2\mathcal{F}\left[f(x)\right]\tag{2.2.17}$$

(3) Shifting on the ω -axis

$$\mathcal{F}^{-1}\left[\hat{f}(\omega-a)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega-a)\,e^{i\omega x}d\omega$$

Let $\omega-a=v$, then $\omega=v+a,\ d\omega=dv$. We obtain

$$\begin{aligned}\mathcal{F}^{-1}\left[\hat{f}(\omega-a)\right]&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(v)\,e^{i(v+a)x}dv=\frac{1}{\sqrt{2\pi}}e^{iax}\int_{-\infty}^{\infty}\hat{f}(v)\,e^{ivx}dv\\&=e^{iax}\mathcal{F}^{-1}\left[\hat{f}(\omega)\right]=e^{iax}f(x)\end{aligned}\tag{2.2.18}$$

(4) Shifting on the x-axis

$$\mathcal{F}\left[f(x-a)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x-a)\,e^{-i\omega x}dx$$

Let $x-a=v$, then $x=v+a,\ dx=dv$. We obtain

$$\begin{aligned}\mathcal{F}\left[f(x-a)\right]&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(v)\,e^{-i\omega(v+a)}dv=\frac{1}{\sqrt{2\pi}}e^{-ia\omega}\int_{-\infty}^{\infty}f(v)\,e^{-i\omega v}dv\\&=e^{-ia\omega}\mathcal{F}\left[f(x)\right]\end{aligned}\tag{2.2.19}$$

(5) Differentiation of the Fourier transform

$$\hat{f}'(\omega)=\frac{d\hat{f}(\omega)}{d\omega}=\frac{d}{d\omega}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-i\omega x}dx\right\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)\frac{de^{-i\omega x}}{d\omega}dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}-ixf(x)e^{-i\omega x}dx$$
$$=-i\mathscr{F}\left[xf(x)\right]$$

Thus,

$$\mathscr{F}\left[xf(x)\right]=i\hat{f}'(\omega) \tag{2.2.20}$$

(6) Scaling on the x-axis

If $a>0$ then

$$\mathscr{F}\left[f(ax)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(ax)e^{-i\omega x}dx$$

Let $ax=v$, we get

$$\mathscr{F}\left[f(ax)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{a}f(v)e^{-i\omega\frac{v}{a}}dv=\frac{1}{a}\hat{f}(\frac{\omega}{a})$$

We can similarly derive it if $a<0$. Hence,

$$\mathscr{F}\left[f(ax)\right]=\frac{1}{|a|}\hat{f}(\frac{\omega}{a}) \tag{2.2.21}$$

(7) Convolution

$$\mathscr{F}\left[(f*g)(x)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(p)g(x-p)e^{-i\omega x}dpdx \tag{2.2.22}$$

Let $x-p=q$, we get

$$\begin{aligned}\mathscr{F}\left[(f*g)(x)\right]&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(p)g(q)e^{-i\omega(p+q)}dqdp\\&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(p)e^{-i\omega p}dp\int_{-\infty}^{\infty}g(q)e^{-i\omega q}dq=\sqrt{2\pi}\mathscr{F}\left[f(x)\right]\mathscr{F}\left[g(x)\right]\end{aligned} \tag{2.2.23}$$

Also,

$$\mathscr{F}\left[f(x)g(x)\right]=\frac{1}{\sqrt{2\pi}}(\hat{f}*\hat{g})(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega-v)\hat{g}(\omega)dv \tag{2.2.24}$$

Note: In general, $\mathscr{F}\left[f(x)g(x)\right]\neq\mathscr{F}\left[f(x)\right]\mathscr{F}\left[g(x)\right]$

(8) Duality

Let $x\to-\omega$ and $\omega\to x$ simultaneously in the inverse Fourier transform. We get

$$f(-\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(x)e^{-i\omega x}dx=\mathscr{F}\left[\hat{f}(x)\right] \tag{2.2.25}$$

Example

Find the real Fourier integral and the Fourier transform of the following function.

$$f(x)=\begin{cases}x & \text{if } 0<x<1\\ 0 & \text{otherwise}\end{cases} \tag{2.2.26}$$

Solution

Substituting Eq.(2.2.26) into Eq.(2.2.5), we obtain

$$\begin{aligned}A(\omega)&=\frac{1}{\pi}\int_{-\infty}^{\infty}f(x)\cos\omega x\,dx=\frac{1}{\pi}\int_{0}^{1}x\cos\omega x\,dx=\frac{-1+\cos\omega+\omega\sin\omega}{\pi\omega^2}\\B(\omega)&=\frac{1}{\pi}\int_{-\infty}^{\infty}f(x)\sin\omega x\,dx=\frac{1}{\pi}\int_{0}^{1}x\sin\omega x\,dx=\frac{\sin\omega-\omega\cos\omega}{\pi\omega^2}\end{aligned} \tag{2.2.27}$$

Hence, we obtain the Fourier integral

$$f(x)=\int_{0}^{\infty}\left[\frac{-1+\cos\omega+\omega\sin\omega}{\pi\omega^2}\cos\omega x+\frac{\sin\omega-\omega\cos\omega}{\pi\omega^2}\sin\omega x\right]d\omega \tag{2.2.28}$$

Substituting Eq.(2.2.26) into Eq.(2.2.9), we obtain the Fourier transform

$$\hat{f}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-i\omega x}dx=\frac{1}{\sqrt{2\pi}}\int_{0}^{1}x\,e^{-i\omega x}dx=\frac{-1+e^{-i\omega}(1+i\omega)}{\sqrt{2\pi}\omega^2} \tag{2.2.29}$$

Figure 2.2.2 shows $A(\omega)$, $B(\omega)$ and $\left|\tilde{f}(\omega)\right| = \sqrt{2/\pi}\left|\hat{f}(\omega)\right|$, which is a frequency domain representation of the function $f(x)$. Figure 2.2.3 shows the integral

$$S_a(x) = \int_0^a \left[\frac{-1+\cos\omega+\omega\sin\omega}{\pi\omega^2}\cos\omega x + \frac{\sin\omega-\omega\cos\omega}{\pi\omega^2}\sin\omega x \right] d\omega \tag{2.2.30}$$

which approximates the integral in Eq.(2.2.28). Although $S_a(x)$ approaches $f(x)$ as a increases, the oscillation known as Gibbs phenomenon occurs near the discontinuity point of $f(x)$.

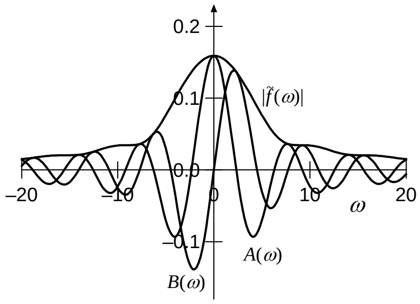


Fig.2.2.2 Amplitude spectrums of $f(x)$

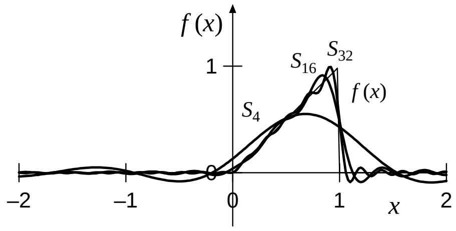


Fig.2.2.3 Integral $S_a(x)$ and $f(x)$

2.2.4 Fourier Transforms of Some Functions

・Gaussian function

$$f(x) = e^{-ax^2} \qquad (a > 0)$$

We use the definition of the Fourier transform.

$$\mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ax^2-i\omega x}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(\sqrt{a}x+\frac{i\omega}{2\sqrt{a}}\right)^2+\left(\frac{i\omega}{2\sqrt{a}}\right)^2} dx = \frac{1}{\sqrt{2\pi}} e^{-\frac{\omega^2}{4a}}\int_{-\infty}^{\infty} e^{-\left(\sqrt{a}x+\frac{i\omega}{2\sqrt{a}}\right)^2} dx$$

We denote the integral by I and we use $\sqrt{a}x + i\omega/2\sqrt{a} = v$ as a new variable of integration. Then $dx = dv/\sqrt{a}$, so that

$$I = \int_{-\infty}^{\infty} e^{-\left(\sqrt{a}x+\frac{i\omega}{2\sqrt{a}}\right)^2} dx = \frac{1}{\sqrt{a}}\int_{-\infty}^{\infty} e^{-v^2} dv$$

We square the integral, convert it to a double integral, and use polar coordinates $r = \sqrt{u^2+v^2}$ and θ . Since $du\,dv = r\,dr\,d\theta$, we get

$$I^2 = \frac{1}{a}\int_{-\infty}^{\infty} e^{-u^2}du\int_{-\infty}^{\infty} e^{-v^2}dv = \frac{1}{a}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(u^2+v^2)}du\,dv$$

$$= \frac{1}{a}\int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,dr\,d\theta = \frac{2\pi}{a}\left[-\frac{1}{2}e^{-r^2}\right]_0^{\infty} = \frac{\pi}{a}$$

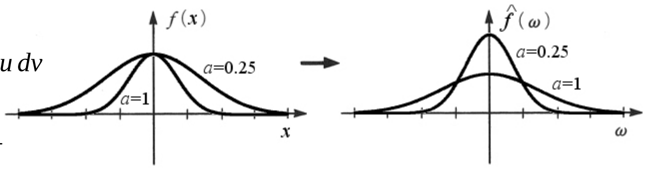


Fig.2.2.4 Gaussian function and its Fourier transform

Hence $I = \sqrt{\pi/a}$.

From this and the first equation in this solution,

$$\mathcal{F}\left[e^{-ax^2}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ax^2-i\omega x}dx = \frac{1}{\sqrt{2\pi}} e^{-\frac{\omega^2}{4a}}\sqrt{\frac{\pi}{a}} = \frac{1}{\sqrt{2a}} e^{-\frac{\omega^2}{4a}} \tag{2.2.31}$$

The Fourier transform of the Gaussian function is a Gaussian function.

・Square wave

$$f_a(x) = \begin{cases} \dfrac{1}{2a} & \text{if } |x| < a \\ 0 & \text{if } |x| > a \end{cases} \tag{2.2.32}$$

Area: $\int_{-\infty}^{\infty} f_a(x)dx = 1$

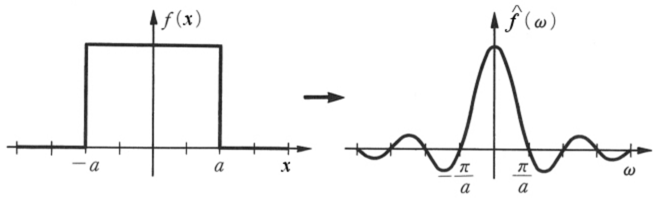


Fig.2.2.5 Square wave and its Fourier transform

Taking the Fourier transform, we get

$$\mathcal{F}\left[f_a(x)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f_a(x)e^{-i\omega x}dx=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}\frac{1}{2a}e^{-i\omega x}dx=\frac{1}{\sqrt{2\pi}}\frac{1}{2a}\left[\frac{-1}{i\omega}e^{-i\omega x}\right]_{-a}^{a}$$

$$=\frac{1}{\sqrt{2\pi}}\frac{1}{2a}\frac{-1}{i\omega}\left(e^{-i\omega a}-e^{i\omega a}\right) \tag{2.2.33}$$

$$=\frac{1}{\sqrt{2\pi}}\frac{1}{2a}\frac{-1}{i\omega}\left(-2i\sin\omega a\right)=\frac{1}{\sqrt{2\pi}}\frac{\sin a\omega}{a\omega}$$

・Dirac delta function (unit impulse function)

$$\lim_{a\to 0}f_a(x)=\delta(x)=\begin{cases}\infty & \text{if } x=0\\ 0 & \text{if } x\neq 0\end{cases} \tag{2.2.34}$$

Properties: $\int_{-\infty}^{\infty}\delta(x)dx=1$

$$\int_{-\infty}^{\infty}f(x)\delta(x-a)dx=f(a) \tag{2.2.35}$$

$$\int_{-\infty}^{\infty}f(x)\delta(x)dx=f(0)$$

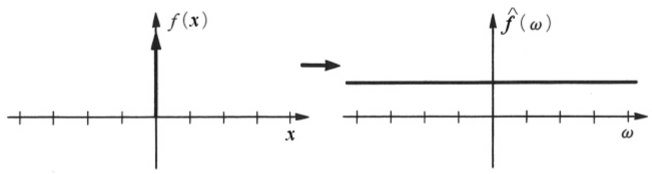


Fig.2.2.6 Dirac delta function $\delta(x)$ and its Fourier transform

Taking the Fourier transform, we get

$$\mathcal{F}\left[\delta(x)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(x)e^{-i\omega x}dx$$

$$=\frac{1}{\sqrt{2\pi}}e^{-i\omega 0}=\frac{1}{\sqrt{2\pi}} \tag{2.2.36}$$

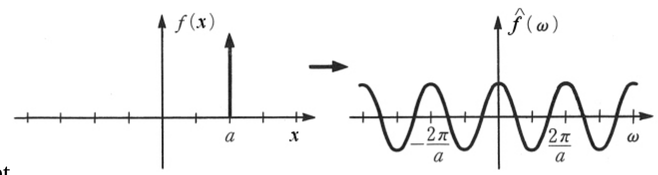


Fig.2.2.7 $\delta(x-a)$ and its Fourier transform

Using the formula of shifting on the x-axis, we get

$$\mathcal{F}\left[\delta(x-a)\right]=e^{-i\omega a}\mathcal{F}\left[\delta(x)\right]$$

$$=\frac{1}{\sqrt{2\pi}}e^{-ia\omega} \tag{2.2.37}$$

$$=\frac{1}{\sqrt{2\pi}}\left(\cos a\omega-i\sin a\omega\right)$$

From the duality, we obtain

$$\mathcal{F}\left[1\right]=\sqrt{2\pi}\delta(-\omega)=\sqrt{2\pi}\delta(\omega) \tag{2.2.38}$$

and

$$\mathcal{F}\left[e^{-iax}\right]=\sqrt{2\pi}\delta(-\omega-a)=\sqrt{2\pi}\delta(\omega+a)$$

$$\mathcal{F}\left[e^{iax}\right]=\sqrt{2\pi}\delta(\omega-a) \tag{2.2.39}$$

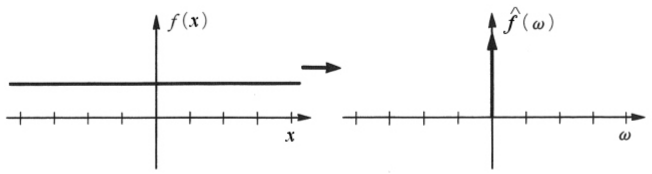


Fig.2.2.8 Constant and its Fourier transform

・Trigonometric function

$$f(x)=a\cos\omega_0 x+b\sin\omega_0 x$$

$$=\frac{1}{2}\left\{(a-ib)e^{i\omega_0 x}+(a+ib)e^{-i\omega_0 x}\right\}$$

where ω_0 is constant.

Taking the Fourier transform, we get

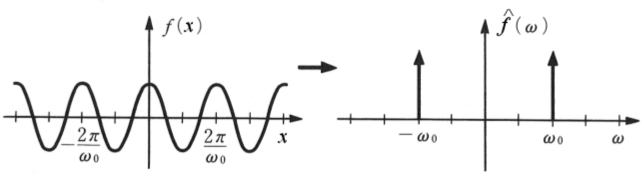


Fig.2.2.9 Trigonometric function and its Fourier transform

$$\mathcal{F}\left[a\cos\omega_0 x+b\sin\omega_0 x\right]=\frac{\sqrt{2\pi}}{2}\left\{(a-ib)\delta(\omega-\omega_0)+(a+ib)\delta(\omega+\omega_0)\right\} \tag{2.2.40}$$

・Impulse train (Dirac comb)

$$\delta_T(x) = \sum_{n=-\infty}^{\infty} \delta(x - nT) \qquad (2.2.41)$$

Because it is a periodic function of period $p = T$, it has a Fourier series.

The complex Fourier coefficients are

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} \delta_T(x) e^{-in\omega_0 x} dx = \frac{1}{T}$$

where $\omega_0 = 2\pi/T$.

Hence we can express the impulse train as

$$\delta_T(x) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{in\omega_0 x} \qquad (2.2.42)$$

Taking the Fourier transform, we get

$$\begin{aligned}\mathcal{F}\left[\delta_T(x)\right] &= \frac{\sqrt{2\pi}}{T}\sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0) \\ &= \frac{\omega_0}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0) \qquad (2.2.43) \\ &= \frac{\omega_0}{\sqrt{2\pi}}\delta_{\omega_0}(\omega)\end{aligned}$$

The Fourier transform of the impulse train is an impulse train.

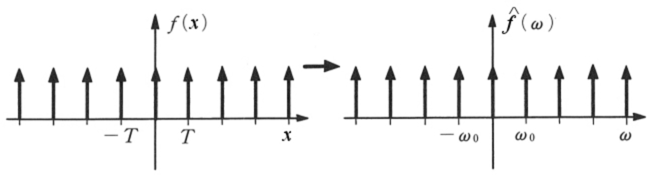


Fig.2.2.10 Impulse train and its Fourier transform

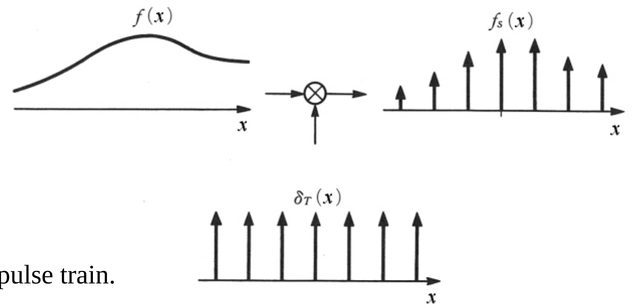


Fig.2.2.11 Property of impulse train

・Periodic function $f(x)$ of period $p = T$

Taking the Fourier transform, we get

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty}\int_{(n-\frac{1}{2})T}^{(n+\frac{1}{2})T} f(x + nT) e^{-i\omega x} dx$$

Let $t = x + nT$, then

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty}\int_{-T/2}^{T/2} f(t) e^{-i\omega(t-nT)} dt = \frac{1}{\sqrt{2\pi}}\left(\sum_{n=-\infty}^{\infty} e^{inT\omega}\right)\int_{-T/2}^{T/2} f(t) e^{-i\omega t} dt$$

Where,

$$\sum_{n=-\infty}^{\infty} e^{inT\omega} = \begin{cases} \infty & \text{if } \omega = \dfrac{2n\pi}{T} \\ 0 & \text{otherwise} \end{cases}$$

From the impulse train, $\displaystyle\sum_{n=-\infty}^{\infty} e^{inT\omega} = \frac{2\pi}{T}\sum_{n=-\infty}^{\infty} \delta(\omega - \frac{2n\pi}{T})$

Furthermore, using the complex Fourier coefficients c_n ,

$$\int_{-T/2}^{T/2} f(t) e^{-i\omega t} dt = Tc_n \quad \text{at} \quad \omega = \frac{2n\pi}{T}$$

Hence we get

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty} Tc_n\omega_0\delta(\omega - n\omega_0) = \sqrt{2\pi}\sum_{n=-\infty}^{\infty} c_n\delta(\omega - n\omega_0) \qquad (2.2.44)$$

where $\omega_0 = 2\pi/T$

The Fourier transform of the periodic function $f(x)$ of period $p = T$ has values only at discrete points $\omega = n\omega_0$, and these values are represented by the complex Fourier coefficients of $f(x)$.

2.2.5 Applications of Fourier Series and Fourier Transforms

・Approximation

Let $f(x)$ be a function on the interval $-\pi \le x \le \pi$ that can be represented on this interval by a Fourier series. Then a **trigonometric polynomial** of degree N

$$F(x) = A_0 + \sum_{n=1}^{N}\left(A_n \cos nx + B_n \sin nx\right) \qquad (N \text{ fixed}) \tag{2.2.45}$$

is an approximation of the given $f(x)$. **The square error** of F in Eq.(2.2.45) (with fixed N) related to the function f on the interval $-\pi \le x \le \pi$

$$E = \int_{-\pi}^{\pi} (f - F)^2 dx \tag{2.2.46}$$

is minimum if and only if the coefficients of F in Eq.(2.2.45) are the Fourier coefficients of f, that is, $A_0 = a_0, A_1 = a_1, \ldots, B_n = b_n$. This minimum value E^* is given by

$$E^* = \int_{-\pi}^{\pi} f^2 dx - \pi\left\{2a_0^2 + \sum_{n=1}^{N}(a_n^2 + b_n^2)\right\} \tag{2.2.47}$$

From Eq.(2.2.47) we see that E^* cannot increase as N increases, but may decrease. Hence, with increasing N**,** the partial sums of the Fourier series of f

$$f(x) \approx a_0 + \sum_{n=1}^{N}\left(a_n \cos nx + b_n \sin nx\right)$$

yield better and better approximations to f, considered from the viewpoint of the square error.

・Frequency analysis

The nature of presentation Eq.(2.2.10) of $f(x)$ becomes clear if we think of it as a superposition of sinusoidal oscillations of all possible frequencies, call a **spectral representation**. In Eq.(2.2.10), the **spectral density** $\hat{f}(\omega)$ measures the intensity of $f(x)$ in the frequency interval between ω and $\omega + \Delta\omega$ ($\Delta\omega$ small, fixed). The integral

$$\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 \, d\omega$$

can be interpreted as the **total energy** of the physical system; hence an integral of $|\hat{f}(\omega)|^2$ from a to b gives the contribution of the frequencies ω between from a to b to the total energy. If a function $f(x)$ is square-integrable, then we have Parseval's theorem

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 \, d\omega \tag{2.2.48}$$

If the system has a periodic solution $y = f(x)$ that can be represented by a Fourier series, then we get a series of squares $|c_n|^2$ of Fourier coefficients c_n given by Eq.(2.1.21). In this case we have a **discrete spectrum** (or **point spectrum**) consisting of countably many isolated frequencies (infinitely many, in general), the corresponding $|c_n|^2$ being the contributions to the total energy.

A system whose solution can be represented by a Fourier integral Eq.(2.2.10) leads to the above integral for the energy.

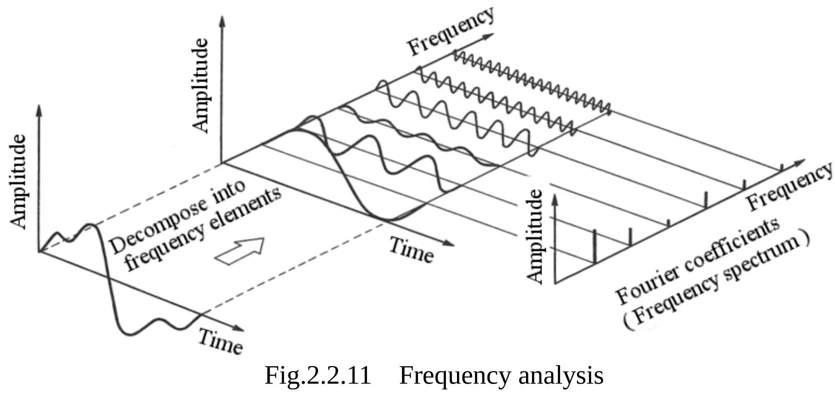


Fig.2.2.11 Frequency analysis

・Solving method of differential equations

In the next section, we will see that the partial differential equations can be solved by the Fourier series or the

Fourier transforms methods.

Problem-1

Find the real Fourier integral and the Fourier transform of the following functions.

(1) $f(x) = x \quad (-1 < x < 1)$

(2) $f(x) = x^2 \quad (-1 < x < 1)$

(3) $f(x) = |x| \quad (-1 < x < 1)$

(4) $f(x) = e^x \quad (-1 < x < 1)$

(5) $f(x) = e^{-|x|}$

(6) $f(x) = e^{-|x|} \sin x$

(7) $f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

(8) $f(x) = \begin{cases} 1 & \text{if } -1 < x < 0 \\ 1 - x & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

(9) $f(x) = \begin{cases} 1 & \text{if } -1/2 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

(10) $f(x) = \begin{cases} 1 + x & \text{if } -1 < x < 0 \\ 1 - x & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

(11) $f(x) = \begin{cases} \sin \pi x & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

(12) $f(x) = \begin{cases} |\sin \pi x| & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

Problem-2

Find the Fourier transform of the following functions.

(1) $f(x) = \sin \pi x$

(2) $f(x) = 4 \cos \pi x + \cos 4\pi x$

2.3 Linear Partial Differential Equations

In this section, we see that the partial differential equations can be solved by the Fourier series or the Fourier transforms methods.

2.3.1 Examples of Linear Partial Differential Equations of the Second Order

Heat equation $$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{2.3.1}$$

Wave equation $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{2.3.2}$$

Laplace equation $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{2.3.3}$$

Linear Homogeneous Partial Differential Equations

The superposition or linearity principle holds.

Example-1

Find the temperature $u(x,t)$ in a bar of length L governed by the following heat equation and the conditions.

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (0 < x < L, t > 0) \tag{2.3.4}$$

Boundary conditions:

$$u(0,t) = 0, \quad u(L,t) = 0 \quad (t \geq 0) \tag{2.3.5}$$

Initial condition:

$$u(x,0) = f(x) = \begin{cases} x & \text{if} \quad 0 < x < L/2 \\ L - x & \text{if} \quad L/2 < x < L \end{cases} \tag{2.3.6}$$

Solution by method of separating variables (review)

We determine solution of the Eq.(2.3.4) of the form

$$u(x,t) = F(x)G(t) \tag{2.3.7}$$

By differentiating and substituting it into Eq.(2.3.4), we obtain

$$F(x)\frac{dG(t)}{dt} = c^2 G(t)\frac{d^2F(x)}{dx^2} \tag{2.3.8}$$

Dividing by $c^2F(x)G(t)$, we find

$$\frac{1}{c^2G(t)}\frac{dG(t)}{dt} = \frac{1}{F(x)}\frac{d^2F(x)}{dx^2} = k \tag{2.3.9}$$

This yields two ordinary differential equations,

$$\frac{d^2F(x)}{dx^2} - kF(x) = 0 \tag{2.3.10}$$

$$\frac{dG(t)}{dt} - kc^2G(t) = 0 \tag{2.3.11}$$

From Eq.(2.3.10) and the boundary conditions, $F(0) = 0$, $F(L) = 0$,

(1) For $k > 0$, $F(x) = Ae^{-\sqrt{k}x} + Be^{\sqrt{k}x}$. From Boundary conditions, $A = B = 0$. Hence $F(x) \equiv 0$

(2) For $k = 0$, $F(x) = ax + b$. From Boundary conditions, $a = b = 0$. Hence $F(x) \equiv 0$

(3) For $k < 0$, $F(x) = A\cos px + B\sin px$, where $p^2 = -k$

From the boundary conditions, $F(0) = A = 0$, $F(L) = B\sin pL = 0$

We must take $B \neq 0$. Hence $\sin pL = 0$, so $p = n\pi/L \quad (n = 1, 2, \cdots)$

Thus, we obtain

$$F(x) = B\sin\frac{n\pi x}{L} \quad (n = 1, 2, \cdots) \tag{2.3.12}$$

For $p^2 = -k$, a solution of Eq.(2.3.11) is

$$G(t) = e^{-c^2p^2t} \tag{2.3.13}$$

The following is omitted.

Solution by Fourier series

We solve the problem using the method of separating variables and the Fourier series. From the boundary conditions, the temperature $u(x,t)$ can be expressed in the form of the Fourier sine series.

$$u(x,t)=\sum_{n=1}^{\infty} G_n(t)\sin\frac{n\pi x}{L} \tag{2.3.14}$$

Substituting Eq.(2.3.14) into Eq.(2.3.4), we obtain

$$\sum_{n=1}^{\infty} G_n'(t)\sin\frac{n\pi x}{L}=-c^2\sum_{n=1}^{\infty} G_n(t)\left(\frac{n\pi}{L}\right)^2\sin\frac{n\pi x}{L} \tag{2.3.15}$$

Hence, $G_n(t)$'s must be satisfy the following equation.

$$G_n'(t)=-c^2\left(\frac{n\pi}{L}\right)^2 G_n(t) \tag{2.3.16}$$

The solution of this equation is

$$G_n(t)=B_n\exp\left(-\frac{c^2n^2\pi^2}{L^2}t\right) \tag{2.3.17}$$

From this and Eq.(2.3.14), we obtain

$$u(x,t)=\sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{L}\exp\left(-\frac{c^2n^2\pi^2}{L^2}t\right) \tag{2.3.18}$$

From the initial condition, we have

$$u(x,0)=f(x)=\sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{L} \tag{2.3.19}$$

This equation represents the Fourier sine expansion of $f(x)$. Hence, B_n 's are the coefficients of the Fourier sine series.

$$B_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \tag{2.3.20}$$

Hence the formal solution is

$$u(x,t)=\frac{2}{L}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\exp\left(-\frac{c^2n^2\pi^2}{L^2}t\right)\int_0^L f(v)\sin\frac{n\pi v}{L}dv \tag{2.3.21}$$

From Eq.(2.3.20), we get

$$B_n=\frac{2}{L}\left\{\int_0^{L/2} x\sin\frac{n\pi x}{L}dx+\int_{L/2}^{L}(L-x)\sin\frac{n\pi x}{L}dx\right\}=\frac{4L}{n^2\pi^2}\sin\frac{n\pi}{2}=\begin{cases}\dfrac{4L}{n^2\pi^2} & (n=1,5,9,\cdots)\\ -\dfrac{4L}{n^2\pi^2} & (n=3,7,11,\cdots)\\ 0 & (n=2,4,6,\cdots)\end{cases} \tag{2.3.22}$$

Hence the solution is

$$u(x,t)=\frac{4L}{\pi^2}\left[\sin\frac{\pi x}{L}\exp\left(\frac{-c^2\pi^2}{L^2}t\right)-\frac{1}{9}\sin\frac{3\pi x}{L}\exp\left(\frac{-9c^2\pi^2}{L^2}t\right)+\cdots\right] \tag{2.3.23}$$

Figure 2.3.1 shows the temperature for $c=1$, $L=2$ and various values of time.

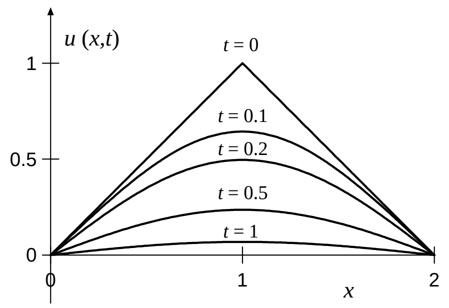


Fig.2.3.1 Temperature in the bar of Length 2

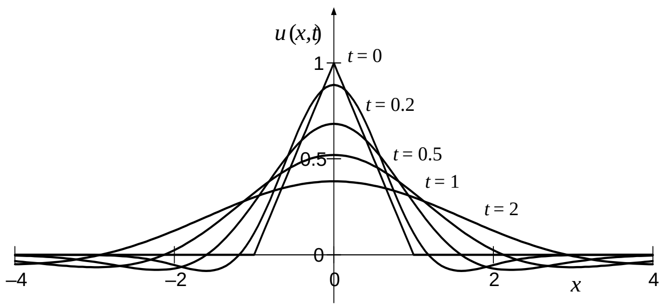


Fig.2.3.2 Temperature in the infinite bar

Example-2

Find the temperature $u(x,t)$ in the infinite bar $(-\infty < x < \infty)$ governed by the following heat equation and the initial temperature.

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (-\infty < x < \infty,\, t > 0) \tag{2.3.24}$$

$$u(x,0) = f(x) = \begin{cases} 1-|x| & \text{if} \quad |x|<1 \\ 0 & \text{if} \quad |x|>1 \end{cases} \tag{2.3.25}$$

Solution

Let the Fourier transforms of $u(x,t)$ and $f(x)$ with respect to x be

$$\hat{u}(\omega,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,t)\,e^{-i\omega x}dx, \quad \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{-i\omega x}dx \tag{2.3.26}$$

The Fourier transforms of Eq.(2.3.24) and Eq.(2.3.25) become

$$\frac{d\hat{u}(\omega,t)}{dt} = c^2 (i\omega)^2 \hat{u}(\omega,t), \quad \hat{u}(\omega,0) = \hat{f}(\omega) \tag{2.3.27}$$

The general solution of this differential equation is

$$\hat{u}(\omega,t) = \tilde{C}\,e^{-c^2\omega^2 t} \tag{2.3.28}$$

From the initial condition, we get

$$\hat{u}(\omega,0) = \tilde{C} = \hat{f}(\omega) \tag{2.3.29}$$

Hence the solution of this initial value problem is

$$\hat{u}(\omega,t) = \hat{f}(\omega)\,e^{-c^2\omega^2 t} \tag{2.3.30}$$

Taking the inverse Fourier transform, we obtain the formal solution

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{u}(\omega,t)\,e^{i\omega x}d\omega = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\omega)\,e^{-c^2\omega^2 t}e^{i\omega x}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(v)\int_{-\infty}^{\infty} e^{-c^2\omega^2 t + i\omega(x-v)}d\omega\,dv = \frac{1}{2c\sqrt{\pi t}}\int_{-\infty}^{\infty} f(v)\exp\left(\frac{-(x-v)^2}{4c^2 t}\right)dv \end{aligned} \tag{2.3.31}$$

The last equation can be obtained by a similar method to Eq.(2.2.31).
Hence the solution is

$$u(x,t)\frac{1}{2c\sqrt{\pi t}}\left[\int_{-1}^{0}(1+v)\exp\left(\frac{-(x-v)^2}{4c^2 t}\right)dv + \int_{0}^{1}(1-v)\exp\left(\frac{-(x-v)^2}{4c^2 t}\right)dv\right] \tag{2.3.32}$$

Figure 2.3.2 shows the temperature for $c=1$ and various values of time.
Taking $z = (v-x)/(2c\sqrt{t})$ as a variable of integration in Eq.(2.3.31), we get the alternative form

$$u(x,t)\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x+2cz\sqrt{t})\,e^{-z^2}\,dz \tag{2.3.33}$$

Example-3

Find the deflection in the string of length of L governed by the following wave equation and the conditions.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (0 < x < L,\, t > 0) \tag{2.3.34}$$

$$u(0,t) = 0, \quad u(L,t) = 0 \quad (t \geq 0) \tag{2.3.35}$$

$$u(x,0) = f(x) = \begin{cases} x & \text{if} \quad 0 < x < L/2 \\ L-x & \text{if} \quad L/2 < x < L \end{cases}, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) = 0 \tag{2.3.36}$$

Solution

Similar to Example 1, from the boundary conditions Eq.(2.3.35),

$$u(x,t) = \sum_{n=1}^{\infty} G_n(t)\sin\frac{n\pi x}{L} \tag{2.3.37}$$

Substituting it into the wave equation Eq.(2.3.34), we obtain

$$\frac{d^2 G_n(t)}{dt^2} = -c^2\left(\frac{n\pi}{L}\right)^2 G_n(t) \tag{2.3.38}$$

Hence,

$$u(x,t)=\sum_{n=1}^{\infty}\left(B_n\cos\frac{cn\pi t}{L}+C_n\sin\frac{cn\pi t}{L}\right)\sin\frac{n\pi x}{L} \tag{2.3.39}$$

From the initial condition Eq.(2.3.36),

$$u(x,0)=f(x) \qquad \text{Fourier sine series} \quad B_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \tag{2.3.40}$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0}=g(x)=0 \qquad \text{Fourier sine series} \quad C_n=\frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}dx=0 \tag{2.3.41}$$

Hence the formal solution is

$$u(x,t)=\sum_{n=1}^{\infty}B_n\cos\frac{cn\pi t}{L}\sin\frac{n\pi x}{L}, \qquad B_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \tag{2.3.42}$$

Example-4

Find the deflection in the infinite string $(-\infty<x<\infty)$ governed by the following wave equation and the conditions.

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2} \quad (-\infty<x<\infty,\ t>0) \tag{2.3.43}$$

$$u(x,0)=f(x)=\begin{cases}1-|x| & \text{if } |x|<1\\ 0 & \text{if } |x|>1\end{cases}, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0}=0 \tag{2.3.44}$$

D'Alembert's solution of the wave equation (review)

By introducing the new independent variables,

$$\xi=x-ct, \quad \eta=x+ct, \tag{2.3.45}$$

a general solution of the wave equation Eq.(2.3.43) is

$$u(x,t)=\varphi(x-ct)+\psi(x+ct) \tag{2.3.46}$$

Then we find the solution satisfying the initial conditions.

Solution by Fourier transform

The Fourier transforms with respect to x of the wave equation Eq.(2.3.43) and the conditions Eq.(2.3.44) become

$$\frac{d^2\hat{u}(\omega,t)}{dt^2}=-c^2\omega^2\hat{u}(\omega,t) \tag{2.3.47}$$

$$\hat{u}(\omega,0)=\hat{f}(\omega), \quad \left.\frac{d\hat{u}(\omega,t)}{dt}\right|_{t=0}=0 \tag{2.3.48}$$

We obtain a general solution of Eq.(2.3.47),

$$\hat{u}(\omega,t)=Ae^{c\omega it}+Be^{-c\omega it} \tag{2.3.49}$$

From the initial conditions Eq.(2.3.48), the solution is

$$\hat{u}(\omega,t)=\frac{1}{2}\hat{f}(\omega)\left(e^{c\omega it}+e^{-c\omega it}\right) \tag{2.3.50}$$

Taking the inverse Fourier transform of Eq.(2.3.50), we obtain the formal solution

$$u(x,t)=\frac{1}{2}\left\{f(x+ct)+f(x-ct)\right\} \tag{2.3.51}$$

Example-5

Find the solution of the following Laplace equation.

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad (0<x<L,\ 0<y<K) \tag{2.3.52}$$

$$u(0,y)=0, \quad u(L,y)=0 \quad (0<y<K) \tag{2.3.53}$$

$$u(x,0)=f(x)=\begin{cases}x & \text{if } 0<x<L/2\\ L-x & \text{if } L/2<x<L\end{cases}, \quad u(x,K)=0 \quad (0<x<L) \tag{2.3.54}$$

Solution

From the boundary conditions, Eq.(2.3.53)

$$u(x,y)=\sum_{n=1}^{\infty}G_n(y)\sin\frac{n\pi x}{L} \tag{2.3.55}$$

Substituting it into the Laplace equation Eq.(2.3.52), we obtain

$$\frac{d^2G_n(y)}{dy^2}=\left(\frac{n\pi}{L}\right)^2 G_n(y) \tag{2.3.56}$$

Hence,

$$G_n(y)=a_n\exp\left(\frac{n\pi y}{L}\right)+b_n\exp\left(-\frac{n\pi y}{L}\right) \tag{2.3.57}$$

From the boundary condition, $u(x,K)=0 \quad (0<x<L)$,

$$G_n(K)=a_n\exp\left(\frac{n\pi K}{L}\right)+b_n\exp\left(-\frac{n\pi K}{L}\right)=0 \tag{2.3.58}$$

We obtain

$$b_n=-a_n\exp\left(\frac{2n\pi K}{L}\right) \tag{2.3.59}$$

Hence,

$$\begin{aligned} G_n(y)&=a_n\left\{\exp\left(\frac{n\pi y}{L}\right)-\exp\left(\frac{2n\pi K-n\pi y}{L}\right)\right\} \\ &=a_n\exp\left(\frac{n\pi K}{L}\right)\left\{\exp\left(\frac{-n\pi(K-y)}{L}\right)-\exp\left(\frac{n\pi(K-y)}{L}\right)\right\} \\ &=2a_n\exp\left(\frac{n\pi K}{L}\right)\sinh\left(\frac{-n\pi(K-y)}{L}\right) \end{aligned} \tag{2.3.60}$$

Let $2a_n\exp\left(\frac{n\pi K}{L}\right)=C_n$ and from the above equation,

$$u(x,y)=\sum_{n=1}^{\infty}C_n\sinh\left(\frac{-n\pi(K-y)}{L}\right)\sin\frac{n\pi x}{L} \tag{2.3.61}$$

From the boundary condition, $u(x,0)=f(x)$,

$$f(x)=\sum_{n=1}^{\infty}C_n\sinh\left(\frac{-n\pi K}{L}\right)\sin\frac{n\pi x}{L},\quad C_n\sinh\left(\frac{-n\pi K}{L}\right)=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx=B_n \tag{2.3.62}$$

Hence the formal solution is

$$u(x,y)=\sum_{n=1}^{\infty}B_n\frac{\sinh\left(\frac{n\pi(K-y)}{L}\right)}{\sinh\left(\frac{n\pi K}{L}\right)}\sin\frac{n\pi x}{L},\quad B_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \tag{2.3.63}$$

Problem

Find the temperature in a bar of length L governed by the following heat equation and the conditions.

$$\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2}\quad (0<x<L,\ t>0) \tag{2.3.64}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0}=0,\quad \left.\frac{\partial u}{\partial x}\right|_{x=L}=0\quad (t\geq 0) \tag{2.3.65}$$

$$u(x,0)=f(x)=\begin{cases} x & \text{if}\quad 0<x<L/2 \\ L-x & \text{if}\quad L/2<x<L \end{cases} \tag{2.3.66}$$

Hint

From the boundary conditions, the temperature $u(x,t)$ can be expressed in the form of the Fourier cosine series.

$$u(x,t)=\sum_{n=0}^{\infty}G_n(t)\cos\frac{n\pi x}{L} \tag{2.3.67}$$