3. Laplace Transforms

The Fourier transforms of the unit step function and the sinusoidal functions don't exist if generalized functions are not used. The Laplace transform corresponds to the Fourier transform multiplied by convergence factor $e^{-\sigma t}, (\sigma > 0)$ and integrated with respect to t from zero to infinity. Although the mathematical concepts behind the Fourier and Laplace transforms are different, we may consider the Fourier transform as a special version of the Laplace transform for real frequencies, i.e. for $s = i\omega$. The Laplace transforms are used in solving ordinary differential equations and initial value problems.

3.1 Laplace Transforms and Inverse Transforms

3.1.1 Laplace Transforms and Inverse Laplace Transforms

Let $f(t)$ be a given function that is defined for all $t \geq 0$. If $\int_0^\infty |f(t)| e^{-\sigma t} dt$ exists and it has some finite value, then it is a function of s, say, $F(s)$:

$$F(s) = \int_0^\infty f(t) e^{-st} dt$$

This function $F(s)$ of the variable s is called the **Laplace Transform** of the original function $f(t)$, and will be denoted by $\mathscr{L}[f(t)]$. Thus

Laplace transform

$$F(s) = \mathscr{L}[f(t)] = \int_0^\infty f(t) e^{-st} dt \qquad (s \text{ is a complex number}) \qquad (3.1.1)$$

Inverse Laplace transform

$$f(t) = \mathscr{L}^{-1}[F(s)] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) e^{st} ds \qquad (c > \sigma) \qquad (3.1.2)$$

(Review) Fourier transform

$$\hat{f}(\omega) = \mathscr{F}[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \qquad (3.1.3)$$

If we set $s = i\omega$ in the Laplace Transform Eq.(3.1.1), it becomes the similar form of the Fourier transform Eq.(3.1.3). If $s = \sigma + i\omega$ (σ and ω are the real number and $\sigma > 0$), it seems that $f(t)$ in Eq.(3.1.3) is multiplied by convergence factor $e^{-\sigma t}$.

Table 3.1.1 Formulas of the Laplace transform (a and b are constants, $u(t-a)$ is the unit step function)

(1)	Linearity	$af(t) + bg(t)$	$aF(s) + bG(s)$
(2)	Derivative of function	$f^{(n)}(t)$	$s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0)$
(3)	Integral of function	$\int_0^t f(\tau) d\tau$	$\frac{F(s)}{s}$
(4)	s-Shifting	$e^{at} f(t)$	$F(s-a)$
(5)	t-Shifting	$f(t-a)u(t-a)$	$e^{-as} F(s)$
(6)	Differentiation of transform	$t\, f(t)$	$-F'(s)$
(7)	Integration of transform	$\frac{f(t)}{t}$	$\int_s^\infty F(\tilde{s}) d\tilde{s}$
(8)	Scaling $(a > 0)$	$f(at)$	$\frac{1}{a} F(s/a)$
(9)	Convolution	$(f * g)(t) = \int_0^t f(\tau) g(t-\tau) d\tau = \int_0^t f(t-\tau) g(\tau) d\tau$	$F(s)G(s)$

Table 3.1.2 Some functions $f(t)$ and their Laplace transforms $F(s)$ (a and ω are constants)

$f(t)$	$F(s)$	$f(t)$	$F(s)$
$\delta(t)$ Dirac delta function	1	e^{at}	$\dfrac{1}{s-a}$
$u(t)$ unit step function, 1	$\dfrac{1}{s}$	$\cos\omega t$	$\dfrac{s}{s^2+\omega^2}$
t^n $(n=0,1,2,\cdots)$	$\dfrac{n!}{s^{n+1}}$	$\sin\omega t$	$\dfrac{\omega}{s^2+\omega^2}$

Sufficient Conditions for Existence of the Laplace Transform

If $f(t)$ is piecewise continuous on every finite interval in the range $t \ge 0$ and satisfies

$$\left|f(t)\right| \le Me^{\gamma t} \quad \text{for all } t \ge 0 \qquad (3.1.4)$$

and for some constants γ and M . Then the Laplace transform of $f(t)$ exists for all $s > \gamma$.

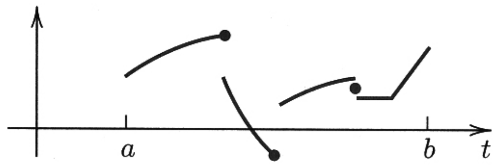


Fig.3.1.1 Example of a piecewise continuous function on an interval $a \le t \le b$

$\because$ Since $f(t)$ is piecewise continuous, $e^{-st}f(t)$ is integrable over any finite interval on the t-axis. From Eq.(3.1.4), assuming that $s > \gamma$, we obtain

$$\left|\mathscr{L}[f(t)]\right| = \left|\int_0^\infty f(t)e^{-st}dt\right| \le \int_0^\infty \left|f(t)\right|e^{-st}dt \le \int_0^\infty Me^{\gamma t}e^{-st}dt = \frac{M}{s-\gamma} \qquad (3.1.5)$$

where the condition $s > \gamma$ was needed for the existence of the last integral.

Uniqueness

If the Laplace transform of a given function exists, it is uniquely determined.

3.1.2 Formulas of the Laplace Transform

(1) Linearity

The Laplace transform is a linear operation; that is, for any functions $f(t)$ and $g(t)$ whose Laplace transforms exist and any constants a and b ,

$$\mathscr{L}\left[af(t)+bg(t)\right] = aF(s)+bG(s) \qquad (3.1.6)$$

$$\because \quad \mathscr{L}\left[af(t)+bg(t)\right] = \int_0^\infty \left\{af(t)+bg(t)\right\}e^{-st}dt = a\int_0^\infty f(t)e^{-st}dt + b\int_0^\infty g(t)e^{-st}dt = aF(s)+bG(s)$$

- Let $f(t)=1$ when $t \ge 0$.

$$\mathscr{L}[1] = \int_0^\infty e^{-st}dt = \left[\frac{-1}{s}e^{-st}\right]_0^\infty . \quad \text{Hence, when } s>0,\ \mathscr{L}[1] = \frac{1}{s} \qquad (3.1.7)$$

- Let $f(t)=t$ when $t \ge 0$.

$$\mathscr{L}[\,t\,] = \int_0^\infty te^{-st}dt = \left[\frac{-1}{s}te^{-st}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}dt = \left[\frac{-1}{s^2}e^{-st}\right]_0^\infty = \frac{1}{s^2} \quad (s>0) \qquad (3.1.8)$$

- It is true for $n=0$ in $\mathscr{L}\left[t^n\right] = n!/s^{n+1}$ because of the above equation and $0!=1$. We now make the induction hypothesis that it hold for any positive integer n .

$$\mathscr{L}\left[t^{n+1}\right] = \int_0^\infty t^{n+1}e^{-st}dt = \left[\frac{-1}{s}t^{n+1}e^{-st}\right]_0^\infty + \frac{n+1}{s}\int_0^\infty t^n e^{-st}dt = \frac{n+1}{s}\mathscr{L}\left[t^n\right] = \frac{n+1}{s}\frac{n!}{s^{n+1}} = \frac{(n+1)!}{s^{n+2}} \qquad (3.1.9)$$

- Let $f(t)=e^{at}$ when $t \ge 0$, where a is a constant.

$$\mathscr{L}\left[e^{at}\right] = \int_0^\infty e^{at}e^{-st}dt = \int_0^\infty e^{-(s-a)t}dt = \left[\frac{-1}{s-a}e^{-(s-a)t}\right]_0^\infty = \frac{1}{s-a} \quad (s-a>0) \qquad (3.1.10)$$

・We set $a = i\omega$ in the above equation. Then

$$\mathscr{L}\left[e^{i\omega t}\right] = \frac{1}{s - i\omega} = \frac{s}{s^2+\omega^2} + i\frac{\omega}{s^2+\omega^2} \tag{3.1.11}$$

On the other hand, by the linearity of the Laplace transform and $e^{i\omega t} = \cos\omega t + i\sin\omega t$,

$$\mathscr{L}\left[e^{i\omega t}\right] = \mathscr{L}[\cos\omega t] + i\mathscr{L}[\sin\omega t] \tag{3.1.12}$$

Equating the real and imaginary parts of these two equations, we obtain

$$\mathscr{L}[\cos\omega t] = \frac{s}{s^2+\omega^2}, \quad \mathscr{L}[\sin\omega t] = \frac{\omega}{s^2+\omega^2} \tag{3.1.13}$$

(2) Laplace transform of the derivative of a function

Suppose that $f(t)$ is continuous for all $t \geq 0$, satisfies Eq.(3.1.4) for some γ and M , and has a derivative $f'(t)$ that is piecewise continuous on every finite interval in the range $t \geq 0$. Then the Laplace transform of the derivative $f'(t)$ exists when $s > \gamma$, and

$$\mathscr{L}\left[f'(t)\right] = s\mathscr{L}\left[f(t)\right] - f(0) = sF(s) - f(0) \quad (s > \gamma) \tag{3.1.14}$$

∵ We first consider the case when $f'(t)$ is continuous for all $t \geq 0$. Then

$$\mathscr{L}\left[f'(t)\right] = \int_0^\infty f'(t)e^{-st}dt = \left[f(t)e^{-st}\right]_0^\infty + s\int_0^\infty f(t)e^{-st}dt$$

For $s > \gamma$, $\mathscr{L}\left[f'(t)\right] = 0 - f(0) + s\mathscr{L}\left[f(t)\right] = sF(s) - f(0)$

If $f'(t)$ is piecewise continuous, the range of integration in the original integral must be broken up into parts such that $f'(t)$ is continuous in each such part.

By applying the above equation to the second derivative $f''(t)$ we obtain

$$\begin{aligned}\mathscr{L}\left[f''(t)\right] &= s\mathscr{L}\left[f'(t)\right] - f'(0) = s\left\{s\mathscr{L}\left[f(t)\right] - f(0)\right\} - f'(0) \\ &= s^2F(s) - sf(0) - f'(0)\end{aligned} \tag{3.1.15}$$

Similarly, we obtain the following extension.

$$\mathscr{L}\left[f^{(n)}(t)\right] = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0) \tag{3.1.16}$$

In Sec.3.2, we will see that the Laplace transform method solves differential equation and corresponding initial and boundary value problems.

Example-1

Let $f(t) = \cos\omega t$. Find $\mathscr{L}\left[f(t)\right]$. (3.1.17)

Solution

$f'(t) = -\omega\sin\omega t$, $f''(t) = -\omega^2\cos\omega t = -\omega^2 f(t)$. Also $f(0) = 1$, $f'(0) = 0$. Then

$$\mathscr{L}\left[f''(t)\right] = -\omega^2\mathscr{L}\left[f(t)\right] = s^2\mathscr{L}\left[f(t)\right] - sf(0) - f'(0) = s^2\mathscr{L}\left[f(t)\right] - s$$

$$(s^2+\omega^2)\mathscr{L}\left[f(t)\right] = s \quad \text{Hence,} \quad \mathscr{L}\left[f(t)\right] = \frac{s}{s^2+\omega^2} \tag{3.1.18}$$

Example-2

Let $f(t) = t\sin\omega t$. Find $\mathscr{L}\left[f(t)\right]$. (3.1.19)

Solution

$f'(t) = \sin\omega t + \omega t\cos\omega t$, $f''(t) = 2\omega\cos\omega t - \omega^2 t\sin\omega t = 2\omega\cos\omega t - \omega^2 f(t)$.

Also, $f(0) = 0$, $f'(0) = 0$. Then

$$\mathscr{L}\left[f''(t)\right] = 2\omega\mathscr{L}[\cos\omega t] - \omega^2\mathscr{L}\left[f(t)\right] = s^2\mathscr{L}\left[f(t)\right] - sf(0) - f'(0) = s^2\mathscr{L}\left[f(t)\right]$$

$$\text{Hence,} \quad \mathscr{L}\left[f(t)\right] = \frac{2\omega}{s^2+\omega^2}\mathscr{L}[\cos\omega t] = \frac{2\omega}{s^2+\omega^2}\frac{s}{s^2+\omega^2} = \frac{2\omega s}{(s^2+\omega^2)^2} \tag{3.1.20}$$

(3) Laplace transform of the integral of a function

If $f(t)$ is piecewise continuous and satisfies Eq.(3.1.4), then

$$\mathscr{L}\left[\int_0^t f(\tau)\,d\tau\right] = \frac{1}{s}\mathscr{L}\left[f(t)\right] = \frac{1}{s}F(s) \quad (s>0, s>\gamma) \tag{3.1.21}$$

∵ Suppose that $f(t)$ is piecewise continuous and satisfies Eq.(3.1.4) for some γ and M. Clearly, if Eq.(3.1.4) holds for some negative γ, it also holds for positive γ, and we may assume that γ is positive. Then the integral

$$g(t) = \int_0^t f(\tau)\,d\tau$$

is continuous, and by using Eq.(3.1.4), we obtain

$$|g(t)| \le \int_0^t |f(\tau)|\,d\tau \le M\int_0^t e^{\gamma\tau}\,d\tau = \frac{M}{\gamma}(e^{\gamma t}-1) \le \frac{M}{\gamma}e^{\gamma t} \qquad (\gamma>0) \tag{3.1.22}$$

This shows that $g(t)$ also satisfies an inequality of the form Eq.(3.1.4). Also, $g'(t) = f(t)$, except for points at which $f(t)$ is discontinuous. Hence $g'(t)$ is piecewise continuous on each finite interval, and, by Eq.(3.1.14),

$$\mathscr{L}\left[f(t)\right] = \mathscr{L}\left[g'(t)\right] = s\mathscr{L}\left[g(t)\right] - g(0) \qquad (s>\gamma) \tag{3.1.23}$$

Here, clearly, $g(0)=0$, so that $\mathscr{L}\left[f(t)\right] = s\mathscr{L}\left[g(t)\right]$. This implies Eq.(3.1.21)

Example-3

Let $\mathscr{L}\left[f(t)\right] = \dfrac{1}{s(s^2+\omega^2)}$. Find $f(t)$.

Solution

From Table 3.1.2 we have

$$\mathscr{L}^{-1}\left[\frac{1}{s^2+\omega^2}\right] = \frac{1}{\omega}\sin\omega t \tag{3.1.24}$$

From this and Eq.(3.1.21), we obtain the answer

$$\mathscr{L}^{-1}\left[\frac{1}{s}\left(\frac{1}{s^2+\omega^2}\right)\right] = \frac{1}{\omega}\int_0^t \sin\omega\tau\,d\tau = \frac{1}{\omega}\left[-\frac{1}{\omega}\cos\omega\tau\right]_0^t = \frac{1}{\omega^2}(1-\cos\omega t) \tag{3.1.25}$$

(4) s -Shifting: Replacing s by $s-a$ in $F(s)$

If $f(t)$ has the Laplace transform $F(s)$ where $s>\gamma$, then

$$\mathscr{L}\left[e^{at}f(t)\right] = F(s-a) \quad (s-a>\gamma) \tag{3.1.26}$$

$$∵\ F(s-a) = \int_0^\infty f(t)e^{-(s-a)t}dt = \int_0^\infty \left\{f(t)e^{at}\right\}e^{-st}dt = \mathscr{L}\left[e^{at}f(t)\right]$$

Example-4

By applying Eq.(3.1.26) to the formulas in Table 3.1.2, we obtain the following results.

$f(t)$	$F(s)$
$e^{at}t^n$	$\dfrac{n!}{(s-a)^{n+1}}$
$e^{at}\cos\omega t$	$\dfrac{s-a}{(s-a)^2+\omega^2}$
$e^{at}\sin\omega t$	$\dfrac{\omega}{(s-a)^2+\omega^2}$

(5) t -Shifting: Replacing t by $t-a$ in $f(t)$

If $f(t)$ has the Laplace transform $F(s)$, then the function

$$\tilde{f}(t)=\begin{cases}0 & \text{if } t<a\\ f(t-a) & \text{if } t>a\end{cases} \qquad (3.1.27)$$

with arbitrary $a \geq 0$ has the Laplace transform $e^{-as}F(s)$.

Using the unit step function

$$u(t-a)=\begin{cases}0 & \text{if } t<a\\ 1 & \text{if } t>a\end{cases} \qquad (3.1.28)$$

we can write $\tilde{f}(t)$ in the form

$$\tilde{f}(t)=f(t-a)u(t-a) \qquad (3.1.29)$$

Then

$$\mathscr{L}\left[f(t-a)u(t-a)\right]=e^{-as}F(s) \qquad (3.1.30)$$

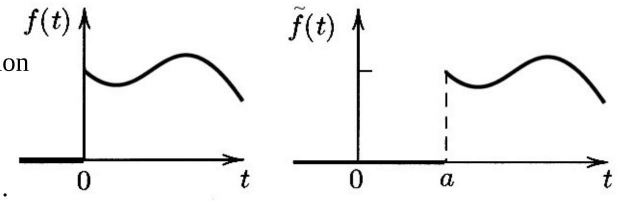


Fig.3.1.2 Function $f(t)$ and $\tilde{f}(t)$

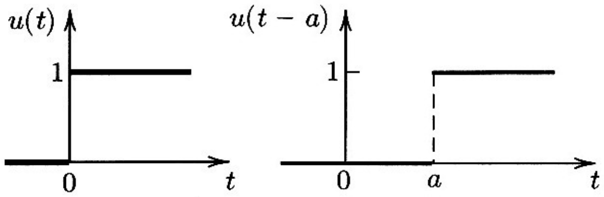


Fig.3.1.3 Unit step function

$\because$ $e^{-as}F(s)=e^{-as}\int_0^{\infty} f(t)e^{-st}dt=\int_0^{\infty} f(t)e^{-s(t+a)}dt$

Substituting $t+a=\tau$ in the integral, we obtain

$$e^{-as}F(s)=\int_a^{\infty} f(\tau-a)e^{-s\tau}d\tau$$

Because $f(\tau-a)u(\tau-a)$ is zero for all τ from 0 to a,

$$e^{-as}F(s)=\int_0^{\infty} f(\tau-a)u(\tau-a)e^{-s\tau}d\tau$$

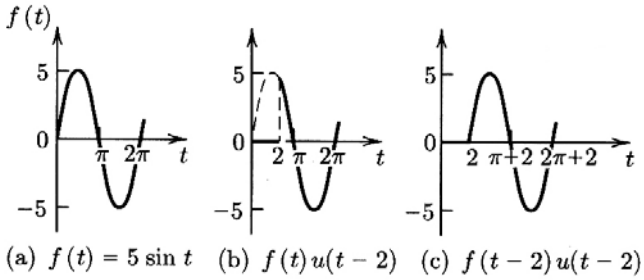


Fig.3.1.4 Effects of the unit step function

・Laplace transform of the unit step function

$$\mathscr{L}\left[u(t-a)\right]=\frac{1}{s}e^{-as}, \mathscr{L}\left[u(t)\right]=\frac{1}{s} \quad (s>0) \qquad (3.1.31)$$

$\because$ $\mathscr{L}\left[u(t-a)\right]=\int_0^{\infty}u(t-a)e^{-st}dt=\int_0^{a}0\cdot e^{-st}dt+\int_a^{\infty}1\cdot e^{-st}dt=\left[\frac{-1}{s}e^{-st}\right]_a^{\infty}=\frac{1}{s}e^{-as}$

Example-5

Find the Laplace transform of the function

$$f(t)=\begin{cases}k & \text{if } 1<t<4\\ -k & \text{if } 4<t<6\\ 0 & \text{otherwise}\end{cases} \qquad (3.1.32)$$

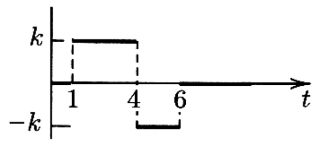
Solution

We write $f(t)$ in the terms of unit step functions.

$$f(t)=k\left[u(t-1)-2u(t-4)+u(t-6)\right] \qquad (3.1.33)$$

Hence we obtain

$$F(s)=k\left(\frac{e^{-s}}{s}-\frac{2e^{-4s}}{s}+\frac{e^{-6s}}{s}\right) \qquad (3.1.34)$$



$k[u(t-1)-2u(t-4)+u(t-6)]$

Fig.3.1.5 Example-5

Example-6

Find the Laplace transform of the function

$$f(t)=\begin{cases}2 & \text{if } 0<t<\pi\\ 0 & \text{if } \pi<t<2\pi\\ \sin t & \text{if } t>2\pi\end{cases} \qquad (3.1.35)$$

Solution

$$\begin{aligned}f(t)&=2u(t)-2u(t-\pi)+u(t-2\pi)\sin t\\ &=2u(t)-2u(t-\pi)+u(t-2\pi)\sin(t-2\pi)\end{aligned} \qquad (3.1.36)$$

$$F(s)=\frac{2}{s}-\frac{2e^{-\pi s}}{s}+\frac{e^{-2\pi s}}{s^2+1} \qquad (3.1.37)$$

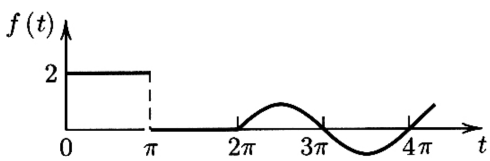


Fig.3.1.6 Example-6

・Single square wave

$$f_{ka}(t) = \begin{cases} \dfrac{1}{k} & \text{if } a \le t \le a+k \\ 0 & \text{otherwise} \end{cases} \quad (3.1.38)$$

This function is represented in the terms of two unit step functions.

$$f_{ka}(t) = \frac{1}{k}\{u(t-a) - u(t-(a+k))\} \quad (3.1.39)$$

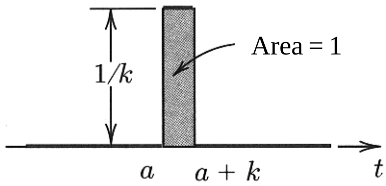


Fig.3.1.7 Single square wave

Taking the Laplace transform, we obtain

$$\mathscr{L}\left[f_{ka}(t)\right] = \frac{1}{k}\left\{\frac{e^{-as}}{s} - \frac{e^{-(a+k)s}}{s}\right\} = e^{-as}\frac{1-e^{-ks}}{ks} \quad (3.1.40)$$

・Dirac delta function (unit impulse function)

$$\delta(t-a) = \lim_{k\to 0} f_{ka}(t)\ .\ \delta(t-a) = \begin{cases} \infty & \text{if } t = a \\ 0 & \text{if } t \ne a \end{cases} \quad (3.1.41)$$

From the above result and l'Hopital's rule, we obtain

$$\mathscr{L}\left[\delta(t-a)\right] = \lim_{k\to 0} e^{-as}\frac{1-e^{-ks}}{ks} = e^{-as}\lim_{k\to 0}\frac{se^{-ks}}{s} = e^{-as} \quad (3.1.42)$$

Also, if $a = 0$, then

$$\mathscr{L}\left[\delta(t)\right] = 1 \quad (3.1.43)$$

(6) Differentiation of transforms

If $f(t)$ satisfies the conditions for the existence of the Laplace transform, then the derivative of its transform with respect to s can be obtained

$$\frac{dF(s)}{ds} = F'(s) = -\int_0^\infty \left[t f(t)\right] e^{-st} dt = \mathscr{L}\left[-t f(t)\right] \quad (3.1.44)$$

$$\because \quad \frac{dF(s)}{ds} = \frac{d}{ds}\int_0^\infty f(t)e^{-st}dt = \int_0^\infty \frac{\partial}{\partial s}\left[f(t)e^{-st}\right]dt = \int_0^\infty -t f(t)e^{-st}dt = \mathscr{L}\left[-t f(t)\right]$$

Differentiation of the transform of a function corresponds to the multiplication of the function by $-t$.

Example-7

Let $f(t) = t\sin\omega t$. Find $\mathscr{L}\left[f(t)\right]$. (3.1.45)

Solution

From Table 3.1.2

$$\mathscr{L}\left[\sin\omega t\right] = \frac{\omega}{s^2+\omega^2} \quad (3.1.46)$$

From this and Eq.(3.1.44), we obtain the answer

$$\mathscr{L}\left[t\sin\omega t\right] = -\frac{d}{ds}\left[\frac{\omega}{s^2+\omega^2}\right] = \frac{2\omega s}{(s^2+\omega^2)^2} \quad (3.1.47)$$

This is identical with Eq.(3.1.20).

Similarly, $\mathscr{L}\left[t\cos\omega t\right] = -\dfrac{d}{ds}\left\{\mathscr{L}\left[\cos\omega t\right]\right\} = -\dfrac{d}{ds}\left[\dfrac{s}{s^2+\omega^2}\right] = \dfrac{s^2-\omega^2}{(s^2+\omega^2)^2}$

(7) Integration of transforms

If $f(t)$ satisfies the conditions for the existence of the Laplace transform and the limit of $f(t)/t$, as t approaches 0 from the right, exists, then

$$\mathscr{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(\tilde{s})d\tilde{s} \qquad (s > \gamma) \quad (3.1.48)$$

$$\because \quad \int_s^\infty F(\tilde{s})d\tilde{s} = \int_s^\infty\left[\int_0^\infty f(t)e^{-\tilde{s}t}dt\right]d\tilde{s} = \int_0^\infty\left[\int_s^\infty f(t)e^{-\tilde{s}t}d\tilde{s}\right]dt = \int_0^\infty f(t)\left[\int_s^\infty e^{-\tilde{s}t}d\tilde{s}\right]dt$$

The integral over $\tilde{s}$ on the right equals e^{-st}/t when $s > \gamma$, and, therefore,

$$\int_s^\infty F(\tilde{s})d\tilde{s} = \int_0^\infty e^{-st}\frac{f(t)}{t}dt = \mathscr{L}\left[\frac{f(t)}{t}\right] \qquad (s > \gamma)$$

Integration of the transform of a function corresponds to the division of the function by t.

(8) Scaling on the t-axis

If $a > 0$ then

$$\mathscr{L}\left[f(at)\right] = \int_0^\infty f(at)e^{-st}dt$$

Let $at = v$ in the integral, we obtain

$$\mathscr{L}\left[f(at)\right] = \int_0^\infty \frac{1}{a} f(v)e^{-s\frac{v}{a}}dv = \frac{1}{a}F(\frac{s}{a}) \qquad (3.1.49)$$

(9) Convolution

Let $f(t)$ and $g(t)$ satisfy the conditions for the existence of the Laplace transform and $\mathscr{L}\left[f(t)\right] = F(s)$, $\mathscr{L}\left[g(t)\right] = G(s)$, then the Laplace transform of the convolution of $f(t)$ and $g(t)$,

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau \qquad (3.1.50)$$

is the product of their Laplace transform, that is,

$$\mathscr{L}\left[(f * g)(t)\right] = F(s)G(s) \qquad (3.1.51)$$

$$\because\ F(s)G(s) = \int_0^\infty f(\tau)e^{-s\tau}d\tau \cdot \int_0^\infty g(p)e^{-sp}dp = \int_0^\infty \int_0^\infty f(\tau)g(p)e^{-s(\tau+p)}d\tau\, dp$$

We now set $t = \tau + p$. The region of this integration is the gray region in Fig.3.1.8.

$$F(s)G(s) = \iint_G f(\tau)g(t-\tau)e^{-st}dt\,d\tau = \int_0^\infty \left[\int_0^t f(\tau)g(t-\tau)\,d\tau\right]e^{-st}dt$$

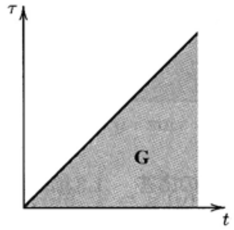


Fig.3.1.8 Region of integration in the $t\tau$ -plane

The transform of a product is generally different from the product of the transforms of the factors,

$$\mathscr{L}\left[f(t)g(t)\right] \neq F(s)G(s) \qquad \text{in general}$$

To see this take $f(t) = e^t$ and $g(t) = 1$. Then $f(t)g(t) = e^t$, $\mathscr{L}\left[f(t)g(t)\right] = 1/(s-1)$, but $\mathscr{L}\left[f(t)\right] = 1/(s-1)$ and $\mathscr{L}\left[1\right] = 1/s$ give $\mathscr{L}\left[f(t)\right]\mathscr{L}\left[g(t)\right] = 1/(s^2 - s)$.

From the definition, it follows almost immediately that convolution has the properties

$$f * g = g * f \qquad \text{(commutative law)} \qquad (3.1.52)$$

$$f * (g_1 + g_2) = f * g_1 + f * g_2 \qquad \text{(distributive law)} \qquad (3.1.53)$$

$$(f * g) * v = f * (g * v) \qquad \text{(associative law)} \qquad (3.1.54)$$

$$f * 0 = 0 * f = 0 \qquad (3.1.55)$$

similar to those of multiplication of numbers. Unusual are the following two properties.

$f * 1 \neq f$ in general. For instance,

$$t * 1 = \int_0^t \tau \cdot 1\,d\tau = \frac{t^2}{2} \neq t$$

$(f * f)(t) \geq 0$ may not hold. For instance,

$$\sin t * \sin t = \int_0^t \sin\tau \sin(t-\tau)\,d\tau = \frac{1}{2}\int_0^t \left\{-\cos t + \cos(2\tau - t)\right\}d\tau$$

$$= \frac{1}{2}\left[-\tau\cos t + \frac{1}{2}\sin(2\tau - t)\right]_0^t = \frac{1}{2}(-t\cos t + \sin t)$$

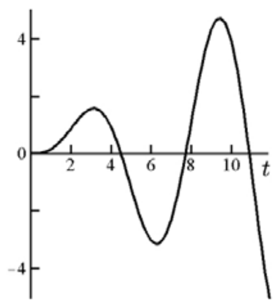


Fig.3.1.9 $\sin t * \sin t$

Example-8

Let $\mathscr{L}\left[f(t)\right] = \dfrac{1}{s(s-a)}$. Find $f(t)$.

Solution

From Table 3.1.2 we know that

$$\mathscr{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}, \quad \mathscr{L}^{-1}\left[\frac{1}{s}\right] = 1$$

Using Eq.(3.1.51), we get the answer

$$e^{at} * 1 = \int_0^t e^{at} \cdot 1\,d\tau = \frac{1}{a}(e^{at} - 1) \qquad (3.1.56)$$

3.1.3 Partial Fractions for the Inverse Laplace Transform

The inverse Laplace transformation is defined as

$$f(t)=\mathscr{L}^{-1}\left[F(s)\right]=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}F(s)e^{st}ds \qquad (\,c>\sigma\,) \tag{3.1.57}$$

The integrand $F(s)e^{st}$ is integrated along a straight line C_1 that is parallel to the imaginary axis on the complex plane as shown in Fig.3.1.10 (a). The line integral is extended to the counterclockwise integration around the simple closed path C that consists of the straight line C_1 and the semicircle C_2 of radius $R\to\infty$ so that all singularities $s_1,\cdots,s_n$ lie inside C as shown in Fig.3.1.10 (b). The integral over the simple closed path is evaluated by using the **residues**.

$$\begin{aligned}\oint_C F(s)e^{st}ds &= \int_{C_1}F(s)e^{st}ds+\int_{C_2}F(s)e^{st}ds \\ &= 2\pi i\sum_{k=1}^{n}\underset{s=s_k}{\mathrm{Res}}\left[F(s)e^{st}\right]\end{aligned} \tag{3.1.58}$$

If $\left|F(s)\right|$ approaches 0 as $|\,s\,|$ approaches infinity, then

$$\int_{C_2}F(s)e^{st}ds=0 \qquad (\,R\to\infty\,,\,t>0\,) \tag{3.1.59}$$

Hence, Eq.(3.1.57) is evaluated as

$$f(t)=\frac{1}{2\pi i}\int_{C_1}F(s)e^{st}ds=\sum_{k=1}^{n}\underset{s=s_k}{\mathrm{Res}}\left[F(s)e^{st}\right] \tag{3.1.60}$$

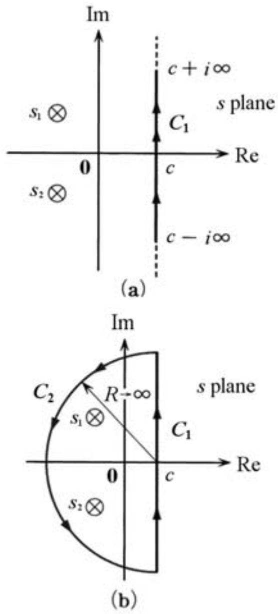


Fig.3.1.10 Path of integration for the inverse Laplace transformation (s_1 and s_2 are singularities)

Example-9

Let $F(s)=\dfrac{s+a}{s(s+1)^2}$. Find $f(t)$.

Solution

The function $F(s)e^{st}$ has a simple pole at $s=0$ and a pole of second order at $s=-1$. These residues are

$$\underset{s=0}{\mathrm{Res}}\left[F(s)e^{st}\right]=\lim_{s\to0}sF(s)e^{st}=\lim_{s\to0}\frac{s+a}{(s+1)^2}e^{st}=a$$

$$\underset{s=-1}{\mathrm{Res}}\left[F(s)e^{st}\right]=\frac{1}{(2-1)!}\lim_{s\to-1}\frac{d}{ds}\left\{(s+1)^2F(s)e^{st}\right\}=\lim_{s\to-1}\frac{d}{ds}\left(\frac{s+a}{s}e^{st}\right)=\lim_{s\to-1}\left(\frac{-a}{s^2}e^{st}+\frac{s+a}{s}te^{st}\right)=-ae^{-t}+(1-a)te^{-t}$$

Hence, we obtain $\quad f(t)=\mathscr{L}^{-1}\left[F(s)\right]=a+(1-a)te^{-t}-ae^{-t}$ (3.1.61)

The inverse of a linear transformation is linear. We can use partial fraction reduction to obtain the inverse Laplace transform. Let $F(s)$ be the following form such as a quotient of two polynomials,

$$F(s)=\frac{B(s)}{A(s)}=\frac{b_ms^m+b_{m-1}s^{m-1}+\cdots+b_1s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0} \qquad (\,n>m\,) \tag{3.1.62}$$

where $A(s)$ and $B(s)$ have real coefficients and no common factors.

(1) If the roots of $A(s)=0$, $\lambda_1,\lambda_2,\cdots,\lambda_n$, are different, then

$$F(s)=\frac{c_1}{s-\lambda_1}+\frac{c_2}{s-\lambda_2}+\cdots+\frac{c_n}{s-\lambda_n}=\sum_{i=1}^{n}\frac{c_i}{s-\lambda_i}\,.\quad c_i=\lim_{s\to\lambda_i}(s-\lambda_i)\frac{B(s)}{A(s)}\quad\text{or}\quad c_i=\frac{B(\lambda_i)}{A'(\lambda_i)} \tag{3.1.63}$$

Its inverse Laplace transform is

$$f(t)=\mathscr{L}^{-1}\left[F(s)\right]=\sum_{i=1}^{n}c_ie^{\lambda_it} \tag{3.1.64}$$

(2) If $A(s)=0$ has multiple roots, that is, λ_1 is a root of order k and $\lambda_{k+1},\cdots,\lambda_n$ are unequal roots, then

$$F(s)=\frac{c_{1k}}{(s-\lambda_1)^k}+\frac{c_{1k-1}}{(s-\lambda_1)^{k-1}}+\cdots+\frac{c_{11}}{s-\lambda_1}+\sum_{i=k+1}^{n}\frac{c_i}{s-\lambda_i}=\sum_{i=1}^{k}\frac{c_{1i}}{(s-\lambda_1)^i}+\sum_{i=k+1}^{n}\frac{c_i}{s-\lambda_i} \tag{3.1.65}$$

$$c_{1i}=\frac{1}{(k-i)!}\lim_{s\to\lambda_1}\frac{d^{k-i}}{ds^{k-i}}\left\{(s-\lambda_i)^k\frac{B(s)}{A(s)}\right\},\quad c_i=\lim_{s\to\lambda_i}(s-\lambda_i)\frac{B(s)}{A(s)} \tag{3.1.66}$$

Its inverse Laplace transform is

$$f(t)=\mathscr{L}^{-1}\left[F(s)\right]=\sum_{i=1}^{k}\frac{c_{1i}}{(i-1)!}t^{i-1}e^{\lambda_1t}+\sum_{i=k+1}^{n}c_ie^{\lambda_it} \tag{3.1.67}$$

(3) If $A(s)=0$ has complex conjugate simple roots, $\lambda_1=\alpha+i\beta$ and $\bar{\lambda}_1=\alpha-i\beta$ (α, β are real numbers), then the corresponding partial fraction is

$$\frac{As+B}{(s-\lambda_1)(s+\bar{\lambda}_1)}=\frac{As+B}{(s-\alpha)^2+\beta^2}=\frac{A(s-\alpha)+(\alpha A+B)}{(s-\alpha)^2+\beta^2} \quad (3.1.68)$$

Its inverse Laplace transform is

$$e^{\alpha t}\left(A\cos\beta t+\frac{\alpha A+B}{\beta}\sin\beta t\right) \quad (3.1.69)$$

where A is the imaginary part and $\frac{\alpha A+B}{\beta}$ the real part of $\frac{1}{\beta}\lim_{s\to\lambda_1}\frac{\left\{(s-\alpha)^2+\beta^2\right\}B(s)}{A(s)}$

Example-10

$$F(s)=\frac{s+5}{s^2-9}=\frac{s+5}{(s+3)(s-3)}=\frac{-1/3}{s+3}+\frac{4/3}{s-3} \qquad f(t)=\mathscr{L}^{-1}\left[F(s)\right]=-\frac{1}{3}e^{-3t}+\frac{4}{3}e^{3t}$$

Example-11

$$F(s)=\frac{s+a}{s(s+1)^2}=\frac{a}{s}+\frac{1-a}{(s+1)^2}+\frac{-a}{s+1} \qquad f(t)=\mathscr{L}^{-1}\left[F(s)\right]=a+(1-a)te^{-t}-ae^{-t}$$

It is identical with the result of Example-9.

Example-12

$$F(s)=\frac{s}{s^2-4s+8}=\frac{s}{(s-2)^2+4}=\frac{(s-2)+2}{(s-2)^2+4} \qquad f(t)=\mathscr{L}^{-1}\left[F(s)\right]=e^{2t}(\cos 2t+\sin 2t)$$

Example-13

$$F(s)=\ln\frac{s^2+16}{s^2}=\ln(s^2+16)-2\ln s$$

Its derivative is

$$F'(s)=\frac{2s}{s^2+16}-\frac{2}{s}$$

Taking the inverse Laplace transform and using the equation of the differentiation of the transform, we obtain

$$\mathscr{L}^{-1}\left[F'(s)\right]=2\cos 4t-2=-tf(t)$$

Hence the inverse Laplace transform of $F(s)$ is

$$\mathscr{L}^{-1}\left[F(s)\right]=f(t)=\frac{2}{t}(-\cos 4t+1) \quad \left(\lim_{t\to 0}f(t)=\lim_{t\to 0}8\sin 4t=0\right)$$

3.1.4 The Laplace Transforms of the Periodic Functions

Let $f(t)$ be a piecewise continuous function that is defined for all positive t and has the period $p\ (>0)$, that is,

$$f(t+np)=f(t) \qquad n=0,\pm1,\pm2,\cdots \qquad \text{for all } t>0 \quad (3.1.70)$$

The Laplace transform of this function is

$$F(s)=\mathscr{L}\left[f(t)\right]=\frac{1}{1-e^{-ps}}\int_0^p f(t)e^{-st}dt \quad (s>0) \quad (3.1.71)$$

∵ We can write the integral from zero to infinity as the series of integrals over successive periods.

$$\mathscr{L}\left[f(t)\right]=\int_0^\infty f(t)e^{-st}dt=\int_0^p e^{-st}f(t)dt+\int_p^{2p}e^{-st}f(t)dt+\int_{2p}^{3p}e^{-st}f(t)dt+\cdots$$

Replacing the variables in each integral,

$$\mathscr{L}\left[f(t)\right]=\int_0^p f(t)e^{-st}dt+\int_0^p e^{-s(\tau+p)}f(\tau+p)d\tau+\int_0^p e^{-s(\tau+2p)}f(\tau+2p)d\tau+\cdots$$

Since $f(t)$ is a periodic function of period p, we thus obtain

$$\mathscr{L}\left[f(t)\right]=\int_0^p f(t)e^{-st}dt+\int_0^p e^{-s(t+p)}f(t)dt+\int_0^p e^{-s(t+2p)}f(t)dt+\cdots$$

$$=\left[1+e^{-sp}+e^{-2sp}+\cdots\right]\int_0^p f(t)e^{-st}dt$$

The series in brackets $[\ldots]$ is a geometric series whose sum is $1/(1-e^{-ps})$. Hence, we obtain

$$\mathscr{L}\left[f(t)\right]=\frac{1}{1-e^{-ps}}\int_0^p f(t)e^{-st}dt \quad (s>0)$$

Example-14

Find the Laplace transforms of the piecewise continuous periodic function of the period $p = 2$ shown in Fig.3.1.11.

$$f(t) = \begin{cases} t & \text{if } 0 < t < 1 \\ 0 & \text{if } 1 < t < 2 \end{cases} \tag{3.1.72}$$

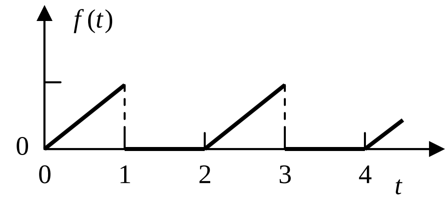


Fig.3.1.11 Example of periodic function

Solution

Since the period $p = 2$, we obtain by integration and simplification

$$\begin{aligned} F(s) &= \frac{1}{1-e^{-ps}}\int_0^p e^{-st} f(t)\,dt = \frac{1}{1-e^{-2s}}\int_0^1 e^{-st} t\,dt \\ &= \frac{1}{1-e^{-2s}}\left(\left[-\frac{t}{s}e^{-st}\right]_0^1 + \frac{1}{s}\int_0^1 e^{-st}dt\right) = \frac{1}{1-e^{-2s}}\left(-\frac{1}{s}e^{-s} - \frac{e^{-s}-1}{s^2}\right) = \frac{1-e^{-s}-se^{-s}}{s^2(1-e^{-2s})} \end{aligned} \tag{3.1.73}$$

3.1.5 Initial Value Theorem and Final Value Theorem

Initial value theorem

$$\lim_{t\to 0+} f(t) = \lim_{s\to\infty} sF(s) \tag{3.1.74}$$

Final value theorem

If all poles of $sF(s)$ are in the left half-plane,

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) \tag{3.1.75}$$

∵ The Laplace transform of the derivative of a function is expressed as follows

$$\int_0^\infty f'(t)e^{-st}dt = sF(s) - f(0) \tag{3.1.76}$$

The left side of Eq.(3.1.76) approaches zero as s approaches infinity. Hence, $0 = \lim_{s\to\infty} sF(s) - f(0)$. This implies Eq.(3.1.74).

If $F(s)$ converges for $s \ge 0$, the left side of Eq.(3.1.76) approaches $\lim_{t\to\infty} f(t) - f(0)$ as s approaches 0. Hence, $\lim_{t\to\infty} f(t) - f(0) = \lim_{s\to 0} sF(s) - f(0)$. This implies Eq.(3.1.75).

The final value theorem is useful because it gives the long-term behavior without having to perform partial fraction decompositions or other difficult algebra. If a function has poles in the right-hand plane or on the imaginary axis, the behavior of this formula is undefined.

Example-15

Let $F(s) = \dfrac{4-s}{s(s+1)(s+2)}$. Find $\lim_{t\to\infty} f(t)$.

Solution

By using the final value theorem, we get

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) = \lim_{s\to 0}\frac{s(4-s)}{s(s+1)(s+2)} = \lim_{s\to 0}\frac{4-s}{(s+1)(s+2)} = 2 \tag{3.1.77}$$

To check it we use the partial fraction expansion

$$F(s) = \frac{4-s}{s(s+1)(s+2)} = \frac{2}{s} - \frac{5}{s+1} + \frac{3}{s+2} \tag{3.1.78}$$

Its inverse Laplace transform is

$$f(t) = \mathscr{L}^{-1}\left[F(s)\right] = 2 - 5e^{-t} + 3e^{-2t} \tag{3.1.79}$$

Hence, $\lim_{t\to\infty} f(t) = 2$.

Example-16: Incorrect use of the final value theorem

Let $F(s) = \dfrac{1}{(s-2)(s+3)}$. Find $\lim_{t\to\infty} f(t)$.

Solution

There exists an unstable pole $s = 2$. The partial fraction expansion gives

$$F(s) = \frac{1}{(s-2)(s+3)} = \frac{1/5}{s-2} - \frac{1/5}{s+3} \tag{3.1.80}$$

Inverse Laplace transformation of $F(s)$ gives

$$f(t) = \mathscr{L}^{-1}[F(s)] = \frac{1}{5}e^{2t} - \frac{1}{5}e^{-3t} \tag{3.1.81}$$

This immediately leads the unbounded final value. But incorrect use of the final value theorem gives the wrong final value as follows.

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{s}{(s-2)(s+3)} = 0 \tag{3.1.82}$$

Problem-1

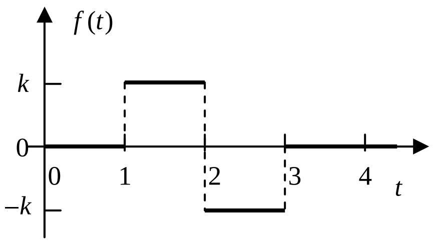
Find the Laplace transforms of the following functions. (ω , θ and k are constants)

(1) $t^2 + 3t - 2$ (2) e^{2t+3} (3) $2te^{5t}$ (4) $t^2 e^{-t/2}$

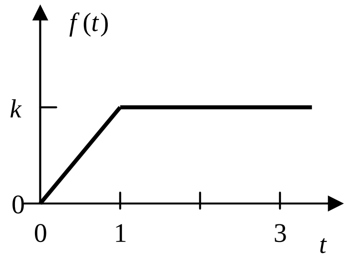
(5) $\cos(\omega t + \theta)$ (6) $\sin^2 3t$ (7) $e^{-2t} \cos 4t$ (8) $t \cos 2t$

(9) $t^2 \cos 3t$ (10) $te^{-2t} \sin 3t$ (11) $t^{-1/2}$ (12) $e^t u(t-2)$

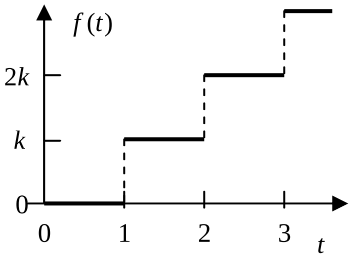
(13)



(14)



(15) Staircase function



Problem-2

Find the Laplace transforms of the following functions, which are assumed to be periodic of the period $p = 2$.

(1) $f(t) = t - 1 \quad (0 < t < 2)$

(2) $f(t) = (t-1)^2 \quad (0 < t < 2)$

(3) $f(t) = |t - 1| \quad (0 < t < 2)$

(4) $f(t) = e^{t-1} \quad (0 < t < 2)$

(5) $f(t) = \begin{cases} 1 & \text{if } 0 < t < 1 \\ 0 & \text{if } 1 < t < 2 \end{cases}$

(6) $f(t) = \begin{cases} t & \text{if } 0 < t < 1 \\ 1 & \text{if } 1 < t < 2 \end{cases}$

(7) $f(t) = \begin{cases} 0 & \text{if } 0 < t < 1/2 \\ 1 & \text{if } 1/2 < t < 2 \end{cases}$

(8) $f(t) = \begin{cases} t & \text{if } 0 < t < 1 \\ 2 - t & \text{if } 1 < t < 2 \end{cases}$

(9) $f(t) = \begin{cases} \sin \pi t & \text{if } 0 < t < 1 \\ 0 & \text{if } 1 < t < 2 \end{cases}$

(10) $f(t) = |\sin \pi t| \quad (0 < x < 2)$

Problem-3

Find the inverse Laplace transforms of the following functions.

(1) $\frac{2}{s-4}$ (2) $\frac{-1}{(s-3)^2}$ (3) $\frac{s+3}{s^2-4}$ (4) $\frac{s+3}{s^2+9}$

(5) $\frac{2s+1}{s^2+4s-5}$ (6) $\frac{s-1}{s^2-4s+5}$ (7) $\frac{s-2}{(s^2+16)^2}$ (8) $\frac{s+1}{s^2(s^2+4)}$

(9) $\frac{s-3}{(s-1)^2(s-4)^2}$ (10) $\ln\frac{s-2}{s+1}$ (11) $\ln\frac{s^2+25}{s^2}$ (12) $\ln\frac{s^2-4}{s^2}$

3.2 Systems of Linear Differential Equations

The Laplace transform method solves differential equation and corresponding initial and boundary value problems.

3.2.1 Initial Value Problem of Systems of Linear Differential Equations with Constant Coefficients

$$\boldsymbol{y}'(t) = \boldsymbol{A}\boldsymbol{y}(t) + \boldsymbol{r}(t) \tag{3.2.1}$$

$$\text{Initial value:} \quad \boldsymbol{y}(0) = \boldsymbol{y}_0 \tag{3.2.2}$$

where $\boldsymbol{y}(t) = [y_1(t) \quad y_2(t) \quad \cdots \quad y_n(t)]^T$, $\boldsymbol{r}(t) = [r_1(t) \quad r_2(t) \quad \cdots \quad r_n(t)]^T$ and $\boldsymbol{A}$ is an $n \times n$ real matrix.

We transform Eq.(3.2.1), writing $\boldsymbol{Y}(s) = \mathscr{L}[\boldsymbol{y}(t)] = \left[\mathscr{L}[y_1(t)] \quad \mathscr{L}[y_2(t)] \quad \cdots \quad \mathscr{L}[y_n(t)]\right]^T$ and $\boldsymbol{R}(s) = \mathscr{L}[\boldsymbol{r}(t)]$ $= \left[\mathscr{L}[r_1(t)] \quad \mathscr{L}[r_2(t)] \quad \cdots \quad \mathscr{L}[r_n(t)]\right]^T$.

This gives

$$s\boldsymbol{Y}(s) - \boldsymbol{y}_0 = \boldsymbol{A}\boldsymbol{Y}(s) + \boldsymbol{R}(s) \tag{3.2.3}$$

Collecting $\boldsymbol{Y}(s)$ -terms, we have

$$\boldsymbol{Y}(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}\{\boldsymbol{y}_0 + \boldsymbol{R}(s)\} \tag{3.2.4}$$

where $\boldsymbol{I}$ is the $n \times n$ unit matrix (identity matrix).

Taking the inverse Laplace transform, we obtain

$$\boldsymbol{y}(t) = \mathscr{L}^{-1}[\boldsymbol{Y}(s)] = \mathscr{L}^{-1}\left[(s\boldsymbol{I} - \boldsymbol{A})^{-1}\{\boldsymbol{y}_0 + \boldsymbol{R}(s)\}\right] \tag{3.2.5}$$

Using the **matrix exponential** defined in Sec.1.2,

$$e^{\boldsymbol{A}t} = \boldsymbol{I} + \boldsymbol{A}t + \frac{1}{2!}\boldsymbol{A}^2t^2 + \cdots + \frac{1}{k!}\boldsymbol{A}^kt^k + \cdots \tag{3.2.6}$$

the solution of the initial value problem is expressed as

$$\boldsymbol{y}(t) = e^{\boldsymbol{A}t}\boldsymbol{y}_0 + \int_0^t e^{\boldsymbol{A}(t-\tau)}\boldsymbol{r}(\tau)d\tau \tag{3.2.7}$$

Hence there is a relation

$$e^{\boldsymbol{A}t} = \mathscr{L}^{-1}\left[(s\boldsymbol{I} - \boldsymbol{A})^{-1}\right] \tag{3.2.8}$$

Table 3.2.1 Properties of matrix exponential $e^{\boldsymbol{A}t}$

(1) If $t = 0$ then $e^{\boldsymbol{A}0} = \boldsymbol{I}$

(2) $\dfrac{d(e^{\boldsymbol{A}t})}{dt} = \boldsymbol{A}e^{\boldsymbol{A}t} = e^{\boldsymbol{A}t}\boldsymbol{A}$

(3) $e^{\boldsymbol{A}t}e^{\boldsymbol{A}\tau} = e^{\boldsymbol{A}(t+\tau)}$

(4) $e^{\boldsymbol{A}t}$ is nonsingular

(5) $(e^{\boldsymbol{A}t})^{-1} = e^{-\boldsymbol{A}t}$

(6) $e^{\boldsymbol{A}t} = \mathscr{L}^{-1}[(s\boldsymbol{I} - \boldsymbol{A})^{-1}]$ $= \mathscr{L}^{-1}\left[\dfrac{\mathrm{adj}(s\boldsymbol{I} - \boldsymbol{A})}{|s\boldsymbol{I} - \boldsymbol{A}|}\right]$

Advantages of the Laplace transform method

(1) Solving a nonhomogeneous ordinary differential equation does not require first solving the homogeneous ordinary differential equation.

(2) Initial values are automatically taken care of.

(3) Complicated input $\boldsymbol{r}(t)$ can be handled very efficiently by using the unit step function $u(t-a)$ and Dirac delta function $\delta(t-a)$.

Example-1

Using Laplace transforms, solve the following initial value problem, which is the same as Eq.(1.2.42) in Sec.1.2.

$$\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y} + \boldsymbol{r}(t) = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}\boldsymbol{y} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}t, \quad \boldsymbol{y}(0) = \boldsymbol{y}_0 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \tag{3.2.9}$$

Solution

By taking the Laplace transform of Eq.(3.2.9), we obtain

$$s\boldsymbol{Y}(s) - \boldsymbol{y}_0 = \boldsymbol{A}\boldsymbol{Y}(s) + \boldsymbol{R}(s) \tag{3.2.10}$$

Solving algebraically, we get

$$\boldsymbol{Y}(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}\{\boldsymbol{y}_0 + \boldsymbol{R}(s)\} = \frac{1}{(s+1)(s+2)}\begin{bmatrix} s+4 & -2 \\ 3 & s-1 \end{bmatrix}\begin{bmatrix} 2 + \dfrac{1}{s^2} \\ 2 + \dfrac{2}{s^2} \end{bmatrix} = \begin{bmatrix} \dfrac{2s^2 + 4s + 1}{s(s+1)(s+2)} \\ \dfrac{2s^3 + 4s^2 + 2s + 1}{s^2(s+1)(s+2)} \end{bmatrix} \tag{3.2.11}$$

Using partial fractions and taking the inverse Laplace transform, we obtain the solution

$$\boldsymbol{y}(t) = \mathscr{L}^{-1}[\boldsymbol{Y}(s)] = \mathscr{L}^{-1}\begin{bmatrix} \dfrac{1}{s+1} + \dfrac{1/2}{s+2} + \dfrac{1/2}{s} \\ \dfrac{1}{s+1} + \dfrac{3/4}{s+2} + \dfrac{1/4}{s} + \dfrac{1/2}{s^2} \end{bmatrix} = \begin{bmatrix} e^{-t} + \dfrac{1}{2}e^{-2t} + \dfrac{1}{2} \\ e^{-t} + \dfrac{3}{4}e^{-2t} + \dfrac{1}{4} + \dfrac{1}{2}t \end{bmatrix} \tag{3.2.12}$$

This is identical with Eq.(1.2.58).

Example-2

Using the Laplace transforms, solve the following initial value problem, which is similar to Eq.(1.2.67) in Sec.1.2.

$$\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y} = \begin{bmatrix} 0 & 1 \\ -25 & -6 \end{bmatrix}\boldsymbol{y}, \quad \boldsymbol{y}(0) = \boldsymbol{y}_0 = \begin{bmatrix} y_{10} \\ y_{20} \end{bmatrix} \tag{3.2.13}$$

Solution

By taking the Laplace transform of Eq.(3.2.13), we obtain

$$s\boldsymbol{Y}(s) - \boldsymbol{y}_0 = \boldsymbol{A}\boldsymbol{Y}(s) \tag{3.2.14}$$

Solving algebraically, we get

$$\boldsymbol{Y}(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{y}_0 = \frac{1}{(s+3)^2 + 4^2}\begin{bmatrix} s+6 & 1 \\ -25 & s \end{bmatrix}\boldsymbol{y}_0 \tag{3.2.15}$$

Taking the inverse Laplace transform, we obtain the solution

$$\boldsymbol{y}(t) = \mathscr{L}^{-1}[(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{y}_0]$$

$$= \mathscr{L}^{-1}\left[\begin{bmatrix} \dfrac{s+3}{(s+3)^2+4^2} + \dfrac{\frac{3}{4}\times 4}{(s+3)^2+4^2} & \dfrac{\frac{1}{4}\times 4}{(s+3)^2+4^2} \\ \dfrac{\frac{-25}{4}\times 4}{(s+3)^2+4^2} & \dfrac{s+3}{(s+3)^2+4^2} + \dfrac{-\frac{3}{4}\times 4}{(s+3)^2+4^2} \end{bmatrix}\begin{bmatrix} y_{10} \\ y_{20} \end{bmatrix}\right] \tag{3.2.16}$$

$$= \begin{bmatrix} e^{-3t}(\cos 4t + \frac{3}{4}\sin 4t) & \frac{1}{4}e^{-3t}\sin 4t \\ -\frac{25}{4}e^{-3t}\sin 4t & e^{-3t}(\cos 4t - \frac{3}{4}\sin 4t) \end{bmatrix}\begin{bmatrix} y_{10} \\ y_{20} \end{bmatrix}$$

This is corresponding to Eq.(1.2.76) with $\tilde{c}_1 = y_{10}, \tilde{c}_2 = (3y_{10} + y_{20})/4$.

Problem-1

Using the Laplace transforms, solve the following initial value problem.

$$\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}, \quad \boldsymbol{y}(0) = \boldsymbol{y}_0$$

(1) $\boldsymbol{A} = \begin{bmatrix} 5 & 4 \\ 2 & 3 \end{bmatrix}, \boldsymbol{y}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (2) $\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}, \boldsymbol{y}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (3) $\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ -5 & -1 \end{bmatrix}, \boldsymbol{y}_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(4) $\boldsymbol{A} = \begin{bmatrix} 1 & 4 \\ -2 & -3 \end{bmatrix}, \boldsymbol{y}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (5) $\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -17 & -8 \end{bmatrix}, \boldsymbol{y}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Problem-2

The following initial value problems are the same problems in Sec.1.2. Solve them by using the Laplace transforms and confirm the answers.

$$\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}, \quad \boldsymbol{y}(0) = \boldsymbol{y}_0$$

(1) $\boldsymbol{A} = \begin{bmatrix} 7 & 5 \\ 1 & 3 \end{bmatrix}, \boldsymbol{y}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (2) $\boldsymbol{A} = \begin{bmatrix} -1 & 2 \\ -2 & -5 \end{bmatrix}, \boldsymbol{y}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (3) $\boldsymbol{A} = \begin{bmatrix} 1 & 5 \\ -2 & -1 \end{bmatrix}, \boldsymbol{y}_0 = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$

(4) $\boldsymbol{A} = \begin{bmatrix} 1 & -2 \\ 4 & -3 \end{bmatrix}, \boldsymbol{y}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (5) $\boldsymbol{A} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, \boldsymbol{y}_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (6) $\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & -2 & -6 \end{bmatrix}, \boldsymbol{y}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

(7) $\boldsymbol{A} = \begin{bmatrix} 0 & 1 & -3 \\ 2 & -1 & -3 \\ 2 & 1 & -5 \end{bmatrix}, \boldsymbol{y}_0 = \begin{bmatrix} y_{10} \\ y_{20} \\ y_{30} \end{bmatrix}$ (8) $\boldsymbol{A} = \begin{bmatrix} 2 & 6 & 4 \\ -4 & -8 & -4 \\ 2 & 3 & 0 \end{bmatrix}, \boldsymbol{y}_0 = \begin{bmatrix} y_{10} \\ y_{20} \\ y_{30} \end{bmatrix}$ (9) $\boldsymbol{A} = \begin{bmatrix} 2 & 6 & 4 \\ -3 & -6 & -2 \\ 1 & 1 & -2 \end{bmatrix}, \boldsymbol{y}_0 = \begin{bmatrix} y_{10} \\ y_{20} \\ y_{30} \end{bmatrix}$